

Representation of the Lagrange reconstructing polynomial by combination of substencils

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Abstract

The Lagrange reconstructing polynomial [Shu C.W.: *SIAM Rev.* **51** (2009) 82–126] of a function $f(x)$ on a given set of equidistant ($\Delta x = \text{const}$) points $\{x_i + \ell \Delta x; \ell \in \{-M_-, \dots, +M_+\}\}$ is defined as the polynomial whose sliding (with x) averages on $[x - \frac{1}{2}\Delta x, x + \frac{1}{2}\Delta x]$ are equal to the Lagrange interpolating polynomial of $f(x)$ on the same stencil [Gerolymos G.A.: *J. Approx. Theory* **163** (2011) 267–305]. We first study the fundamental functions of Lagrange reconstruction, show that these polynomials have only real and distinct roots, which are never located at the cell-interfaces (half-points) $x_i + n\frac{1}{2}\Delta x$ ($n \in \mathbb{Z}$), and obtain several identities. Using these identities, we show that there exists a unique representation of the Lagrange reconstructing polynomial on $\{i - M_-, \dots, i + M_+\}$ as a combination of the Lagrange reconstructing polynomials on Neville substencils [Carlini E., Ferretti R., Russo G.: *SIAM J. Sci. Comp.* **27** (2005) 1071–1091], with weights which are rational functions of ξ ($x = x_i + \xi \Delta x$) [Liu Y.Y., Shu C.W., Zhang M.P.: *Acta Math. Appl. Sinica* **25** (2009) 503–538], and give an analytical recursive expression of the weight-functions. We show that all of the poles of the rational weight-functions are real, and that there can be no poles at half-points. We then use the analytical expression of the weight-functions, combined with the factorization of the fundamental functions of Lagrange reconstruction, to obtain a formal proof of convexity (positivity of the weight-functions) in the neighborhood of $\xi = \frac{1}{2}$, iff all of the substencils contain either point i or point $i + 1$ (or both).

Keywords: reconstruction, (Lagrangian) interpolation and reconstruction, hyperbolic PDEs, finite differences, finite volumes

2000 MSC: 65D99, 65D05, 65D25, 65M06, 65M08

1. Introduction

Polynomial interpolation and/or polynomial reconstruction are the basic numerical approximation operations involved in the development of WENO schemes [1, 2], which are widely used [3] for the discretization of (hyperbolic) PDEs, particularly when the solution contains discontinuities. Following Godunov's theorem [4], these schemes introduce nonlinearity in the approximation (with respect to the reconstructed function $h(x)$ or to its cell-averages $f(x)$), to combine high-order with monotonicity. Central to the development of these methods [5, 3] is the underlying linear approximation, where the interpolating [6, 3] and/or the reconstructing [5, 3] polynomial on a given stencil is represented by a combination of the corresponding (interpolating or reconstructing) polynomials on substencils. We introduce the following definitions

Definition 1.1 (Stencil [7, Definition 4.1, p. 283]). Consider a 1-D homogeneous computational mesh

$$x_i = x_1 + (i - 1)\Delta x \quad \Delta x = \text{const} \in \mathbb{R}_{>0} \quad (1a)$$

Assume

$$M := M_- + M_+ \geq 0 \quad (1b)$$

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The set of contiguous points

$$S_{i,M_-,M_+} := \{i - M_-, \dots, i + M_+\} \quad (1c)$$

is defined as the discretization-stencil in the neighborhood of i , with M_- neighbors to the left and M_+ neighbors to the right. The stencil S_{i,M_-,M_+} (1c) contains $M + 1 > 0$ points and has a length of M intervals. If $M_{\pm} \geq 0$ then the stencil contains the pivot-point i . If $M_-M_+ < 0$ then the stencil does not contain the pivot-point i . We will note

$$[S_{i,M_-,M_+}] := [x_{i-M_-}, x_{i+M_+}] \subset \mathbb{R} \quad (1d)$$

the interval defined by the edge-points of the stencil. \square

Definition 1.2 (Neville substencils). Let S_{i,M_-,M_+} be a discretization stencil on a homogeneous grid (Definition 1.1) with

$$M := M_- + M_+ \geq 2 \quad (2a)$$

Assume

$$K_s \leq M - 1 \quad ; \quad K_s \in \mathbb{N}_0 \quad (2b)$$

The $K_s + 1 \geq 1$ substencils

$$S_{i,M_-,k_s,M_+-K_s+k_s} := \{i - M_- + k_s, \dots, i + M_+ - K_s + k_s\} \quad \forall k_s \in \{0, \dots, K_s\} \quad (2c)$$

each of which contains $M - K_s + 1$ points and which satisfy

$$\bigcup_{k_s=0}^{K_s} S_{i,M_-,k_s,M_+-K_s+k_s} = S_{i,M_-,M_+} \quad (2d)$$

$$\ell_s \neq m_s \iff \begin{cases} S_{i,M_-, \ell_s, M_+-K_s+\ell_s} \not\subset S_{i,M_-, m_s, M_+-K_s+m_s} \\ S_{i,M_-, \ell_s, M_+-K_s+\ell_s} \neq S_{i,M_-, m_s, M_+-K_s+m_s} \end{cases} \quad \forall \ell_s, m_s \in \{0, \dots, K_s\} \quad (2e)$$

$$S_{i-M_-+k_s+1, i+M_+-K_s+k_s+1} = (S_{i-M_-,k_s, i+M_+-K_s+k_s} \setminus \{i - M_- + k_s\}) \cup \{i + M_+ - K_s + k_s + 1\} \quad \forall k_s \in \{0, \dots, K_s - 1\} \quad (2f)$$

are the $(M - K_s + 1)$ -order¹ substencils of S_{i,M_-,M_+} , corresponding to the K_s -level subdivision of S_{i,M_-,M_+} . \square

Definition 1.3 (Reconstruction pair [7, Definition 2.1, p. 270]). Assume that $\Delta x \in \mathbb{R}_{>0}$ is a constant length, and that the functions $f : I \rightarrow \mathbb{R}$ and $h : I \rightarrow \mathbb{R}$ are defined on the interval $I = [a - \frac{1}{2}\Delta x, b + \frac{1}{2}\Delta x] \subset \mathbb{R}$, satisfying everywhere

$$f(x) = \frac{1}{\Delta x} \int_{x-\frac{1}{2}\Delta x}^{x+\frac{1}{2}\Delta x} h(\zeta) d\zeta \quad \forall x \in [a, b] \quad (3a)$$

assuming the existence of the integral in (3a). We will note the functions $f(x)$ and $h(x)$ related by (3a)

$$h = R_{(1;\Delta x)}(f) \quad (3b)$$

$$f = R_{(1;\Delta x)}^{-1}(h) \quad (3c)$$

and will call f and h a reconstruction pair on $[a, b]$, in view of the computation of the 1-derivative.² \square

¹In the sense that the Lagrange interpolating and reconstructing polynomials on each of the substencils (2c) are $O(\Delta x^{M-K_s+1})$ -accurate approximations [7, Proposition 4.6, p. 289].

² By [7, Lemma 2.2, p. 271], (3a) $\implies f^{(n)}(x) = \frac{h^{(n-1)}(x + \frac{1}{2}\Delta x) - h^{(n-1)}(x - \frac{1}{2}\Delta x)}{\Delta x} \quad \forall x \in [a, b] \quad \forall n \in \{1, \dots, N\}$, exactly, assuming $f(x)$ and $h(x)$ are of class $C^N[a - \frac{1}{2}\Delta x, b + \frac{1}{2}\Delta x]$.

Definition 1.4 (Lagrange reconstructing polynomial [7, Definition 2.3, p. 271]). Let $p_{I,M_-,M_+}(x_i + \xi\Delta x; x_i, \Delta x; f)$ denote the Lagrange interpolating polynomial [8, pp. 186–189] of the real function $f : \mathbb{R} \rightarrow \mathbb{R}$ on the stencil S_{i,M_-,M_+} (Definition 1.1). Its reconstruction pair (Definition 1.3)

$$p_{R_1,M_-,M_+}(x_i + \xi\Delta x; x_i, \Delta x; f) := [R_{(1;\Delta x)}(p_{I,M_-,M_+})](x_i + \xi\Delta x; x_i, \Delta x; f) \quad (4)$$

will be called the Lagrange reconstructing polynomial on the stencil S_{i,M_-,M_+} . \square

We study representations where the polynomial approximation on S_{i,M_-,M_+} (Definition 1.1) is expressed as a weighted sum of the corresponding polynomial approximations on the $K_s + 1$ substencils (Definition 1.2)

$$p_{R_1,M_-,M_+}(x_i + \xi\Delta x; x_i, \Delta x; f) = \sum_{k_s=0}^{K_s} \sigma_{R_1,M_-,M_+,K_s,k_s}(\xi) p_{R_1,M_-,M_+-k_s,M_+-K_s+k_s}(x_i + \xi\Delta x; x_i, \Delta x; f) \quad (5a)$$

$$p_{I,M_-,M_+}(x_i + \xi\Delta x; x_i, \Delta x; f) = \sum_{k_s=0}^{K_s} \sigma_{I,M_-,M_+,K_s,k_s}(\xi) p_{I,M_-,M_+-k_s,M_+-K_s+k_s}(x_i + \xi\Delta x; x_i, \Delta x; f) \quad (5b)$$

with weight-functions ($\sigma_{I,M_-,M_+,K_s,k_s}(\xi)$ in the interpolation case or $\sigma_{R_1,M_-,M_+,K_s,k_s}(\xi)$ in the reconstruction case) which are independent of the approximated function ($f(x)$ in the interpolation case or $h(x)$ in the reconstruction case). The subscripts M_{\pm} , K_s and k_s in (5) indicate that the weight-functions depend on the stencil (M_{\pm}), on the level of subdivision (K_s) and on the particular substencil (k_s). Because the weight-functions are independent of the approximated function ($f(x)$ or $h(x)$) they are usually called *linear weights* [3]. Alternatively, since the weights combine the interpolating (or reconstructing) polynomials on the substencils to exactly the interpolating (or reconstructing) polynomial on the entire stencil, they recover the highest possible accuracy (between weighted combinations of the substencils) and, for this reason, they are alternatively called *optimal* (in the sense of accuracy) weights [2, 9].

The underlying linear interpolation or reconstruction used in WENO [5, 3] schemes on the general stencil $\{i - M_-, \dots, i + M_+\}$ (Definition 1.1) can be obtained by writing the approximation error [7, (56a), p. 292] for the $K_s + 1$ ($k_s \in \{0, \dots, K_s\}$) substencils $\{i - M_- + k_s, \dots, i + M_+ - K_s + k_s\}$ (Definition 1.1), each of which has an error of $O(\Delta x^{M-K_s+1})$ [7, Proposition 4.7, p. 292]. At any fixed point $x_i + \xi\Delta x$, we can in this way construct a $(K_s + 1) \times (K_s + 1)$ linear system (eg [10, (13), p. 8489]) for the weights which linearly combine the approximated values on the substencils to obtain an $O(\Delta x^{M+1})$ -accurate approximation, recovering the accuracy (and indeed the exact value [5, 3]) of the entire stencil $\{i - M_-, \dots, i + M_+\}$, at the chosen fixed point $x_i + \xi\Delta x$. It is known by numerical experiment [5, 3], that, for stencils symmetric around x_i (ie $M_- = M_+$) these linear or optimal weights for the $(K_s = M_- = M_+)$ -subdivision, can be calculated at the fixed point $\xi = \frac{1}{2}$ (eg [10, Tab. 3, p. 8484]), ie for this choice of $\{M_{\pm}, K_s, \xi\}$ the linear system [10, (13), p. 8489] is not singular. Shu [5] has given examples of other choices of $\{M_{\pm}, K_s, \xi\}$ for which the linear system is singular. Obviously the weights are functions of ξ , parametrized by $\{M_{\pm}, K_s, k_s\}$.

The Neville-Aitken algorithm [8, pp. 204–209] constructs the interpolating polynomial on $\{i - M_-, \dots, i + M_+\}$, by recursive combination of the interpolating polynomials on substencils, with weights which are also polynomials of x [8, pp. 204–209]. Carlini et al. [6], working on the Lagrange interpolating polynomial in the context of centered (central) WENO schemes, recognized the connexion between the Neville-Aitken algorithm [8, pp. 204–209] and the determination of the optimal weights, and gave the explicit expression [6, (3.6,4.10), pp. 1074–1079] of the polynomial weight-functions $\sigma_{I,r-1,r,r-1,k_s}(\xi)$ which combine the Lagrange interpolating polynomials on the $K_s + 1 = (r-1) + 1$ substencils $\{i - (r-1) + k_s, \dots, i + r - (r-1) + k_s\}$ to obtain the Lagrange interpolating polynomial on the big stencil $\{i - (r-1), \dots, i + r\}$ which contains an odd number of $M = 2r - 1$ intervals and an even number of $M + 1 = 2r$ points. This result was also confirmed by Liu et al. [11, (2.2), p. 506] who further gave the analytical expression [11, (2.18), p. 511] for the polynomial weight-functions $\sigma_{I,r,r,r,k_s}(\xi)$ which combine the Lagrange interpolating polynomials on the $K_s + 1 = r + 1$ substencils $\{i - r + k_s, \dots, i + r - r + k_s\}$ to obtain the Lagrange interpolating polynomial on the big stencil $\{i - r, \dots, i + r\}$ which contains an even number of $M = 2r$ intervals and an odd number of $M + 1 = 2r + 1$ points. For both cases it is shown [6, 11] that $\forall \xi \in [-1, 1]$ the linear weights are positive, and as a consequence the above combination of substencils is convex $\forall \xi \in [-1, 1]$. In a recent work [12] we extended these results for the general K_s -level subdivision of an arbitrary stencil $X_{i-M_-,i+M_+} := \{x_{i-M_-}, \dots, x_{i+M_+}\} \subset \mathbb{R}$ of $M + 1 := M_- + M_+ + 1$ distinct

ordered points on an inhomogeneous grid to $K_s + 1 \leq M$ substencils $X_{i-M_-+k_s, i+M_+-K_s+k_s} := \{x_{i-M_-+k_s}, \dots, x_{i+M_+-K_s+k_s}\}$ ($k_s \in \{0, \dots, K_s\}$), and used a general recurrence relation [12, (4e), Lemma 2.1] to obtain a simplified expression [12, Proposition 3.1] of the weight-functions for the Lagrange interpolating polynomial (5b), and to prove positivity in the interval $x \in [x_{i-M_-+K_s-1}, x_{i+M_+-K_s+1}]$ which contains at least 1 cell (at least 2 grid-points) iff $K_s \leq \left\lceil \frac{M}{2} \right\rceil$ [12, Proposition 3.2].

Looking more carefully into (5b) one notices that it is directly related to Mühlbach's theorem [13, Theorem 2.1, p. 100], corresponding to [13, (2.2,2.3), p. 100]. Mühlbach [13] expresses the coefficient $\sigma_{I, M_-, M_+, K_s, k_s}(\xi)$ (5b) in terms of quotients of determinants of interpolation-error functions, directly obtained by the Cramer solution [14, Proposition 5.1.1, p. 72] of error-eliminating linear systems. Mühlbach [13] studies Chebyshev-systems satisfying interpolatory conditions. In the reconstructing polynomial case (5a), the usual linear system approach [10, (13), p. 8489] is equivalent to the algorithm of Mühlbach [13, Theorem 2.1, p. 100] with the important difference that in (5a) we study polynomials $p_{R_1}(x)$ whose linear functionals $p_l(x) = [R_{(1;\Delta x)}^{-1}(p_{R_1})](x)$ (Definition 1.3) satisfy interpolatory conditions, so that the existence and uniqueness proofs in [13, Theorem 2.1, p. 100] are not directly applicable. Nonetheless, the general recurrence relation for weight-functions proven in [12, (4e), Lemma 2.1], only requires that the $(K_s = 1)$ -level subdivision can be defined. Therefore, finding a general expression for the weight-functions $\sigma_{R_1, M_-, M_+, K_s, k_s}(\xi)$ (5a), is tantamount to solving the problem of the $(K_s = 1)$ -level subdivision for the Lagrange reconstructing polynomial.

Although the reconstructing polynomial [5, 3, 7] is even more widely used in WENO discretizations, the development of practical WENO schemes [2, 9, 10], invariably followed the aforementioned linear system approach [10, (13), p. 8489], using symbolic calculation. There is little analytical work on the weight-functions $\sigma_{R_1, M_-, M_+, K_s, k_s}(\xi)$ which combine the Lagrange reconstructing polynomials on the $K_s + 1 \leq M$ substencils $\{i - M_- + k_s, \dots, i + M_+ - K_s + k_s\}$ to obtain the Lagrange reconstructing polynomial on the big stencil $\{i - M_-, \dots, i + M_+\}$. Only recently, Liu et al. [11] studied particular families of stencils and subdivisions, using symbolic computation.³ Liu et al. [11] have concentrated on the usual WENO substencils.⁴ In the reconstruction case, it was shown by construction [11] that the optimal weight-functions are not polynomials, as in the interpolation case [6, 11, 12], but, instead, rational functions of ξ ($x = x_i + \xi \Delta x$), implying that at the poles of these rational functions the weight-functions cannot be defined. For upwind-biased schemes [2, 9, 10] the big stencil $\{i - (r - 1), \dots, i + (r - 1)\}$ ($r \in \mathbb{N}_{\geq 2}$) which is centered around the point i , and upwind-biased with respect to the cell-face $i + \frac{1}{2}$, is subdivided [11, Tab. 3.2, p. 516] into $K_s + 1 = r$ substencils $\{i - (r - 1) + k_s, \dots, i + k_s\}$, $k_s \in \{0, \dots, K_s\}$. For centered schemes, the big stencil $\{i - (r - 1), \dots, i + r\}$ which is centered with respect to the cell-face $i + \frac{1}{2}$ [5, 15, 3], and as a consequence downwind-biased with respect to the point i , is subdivided into $K_s + 1 = r + 1$ substencils $\{i - (r - 1) + k_s, \dots, i + k_s\}$, $k_s \in \{0, \dots, K_s\}$. In [11, (3.2–3.4), p. 514] an algorithm is sketched for computing the rational weight-functions, which are tabulated up to $r = 7$ [11, Tab. 3.2, p. 516] for the upwind-biased case (even number of intervals) and up to $r = 6$ [11, Tab. 3.5, p. 518] for the centered case (odd number of intervals). We remarked in [7, p. 298] that both these families can be grouped together as the subdivision of the general stencil $\{i - \lfloor \frac{M}{2} \rfloor, \dots, i + M - \lfloor \frac{M}{2} \rfloor\}$ into $K_s + 1 = \left\lceil \frac{M}{2} \right\rceil + 1$ substencils, in the range $M \in \{2 \dots, 11\}$. These weights were further analyzed to determine the regions of convexity of the representation (positivity of the weight-functions). These important results [11] include explicit expressions of the weight-functions for the particular stencils which were studied, but a general analytical expression of the optimal weight-functions for the representation of the Lagrange reconstructing polynomial by combination of substencils is not yet available, contrary to the interpolating polynomial case [6, 11, 12]. The work of Liu et al. [11] is based on the reconstruction via primitive approach [16, pp. 243–244], as developed in [5, 3], where the integral (primitive) $\int_0^x h(\zeta) d\zeta$ of the function $h(x)$, which is reconstructed from its sliding averages $f(x)$ (Definition 1.3), is used.

Despite the enormous successes of the reconstruction via primitive approach [16, pp. 243–244] in designing and analyzing practical WENO schemes [5, 3, 11] the reconstruction via deconvolution approach [16, 244–246] is conceptually more straightforward, since it directly uses the unknown function which is reconstructed from cell-averages, and sometimes simplifies analytical work. In a recent work [7, Lemma 2.5, p. 272] we have provided the analytical solution of the deconvolution problem [16, (3.13b), p. 244], which expresses the unknown function $h(x)$,

³ Liu et al. [11] have examined, using symbolic calculation, the computation and positivity of linear (optimal) weight-functions in WENO interpolation, reconstruction and integration.

⁴ In the nomenclature of Shu [5, 3], used in Liu et al. [11], stencils are defined in terms of cell-interfaces (half-points), and the term nodes in [11] denotes cells, so that the stencil $\{i - (r - 1), \dots, i + (r - 1)\}$ is defined in [11, Tab. 3.2, p. 516] as $\{i - r + \frac{1}{2}, \dots, i + r - \frac{1}{2}\}$, and the stencil $\{i - (r - 1), \dots, i + r\}$ is defined in [11, Tab. 3.5, p. 518] as $\{i - r + \frac{1}{2}, \dots, i + r + \frac{1}{2}\}$

which is reconstructed from its sliding averages, as a series of the derivatives of the sliding averages $f^{(n)}(x)$ [7, (10b), p. 272]. This analytical solution of the deconvolution problem [16, (3.13b), p. 244] allows the analytical computation of the approximation error of the Lagrange reconstructing polynomial [7, Proposition 4.7, p. 292], which would have been necessary to build the general linear system [10, (13), p. 8489] for the weight-functions $\sigma_{R_1, M_-, M_+, K_s, k_s}(\xi)$ in (5a).

In the present work⁵ we use relations and concepts developed in [7], along with the general recurrence relation for the generation of weight-functions proven in [12, Lemma 2.1], to extend the analysis of Liu et al. [11], both by providing general analytical expressions (and existence and uniqueness proofs) of the rational weight-functions, but also by studying the general case of the subdivision of an arbitrarily biased stencil on a homogeneous grid, $\{i - M_-, \dots, i + M_+\}$ ($M_{\pm} \in \mathbb{Z} : M := M_- + M_+ \geq 2$) containing M intervals, into $K_s + 1 \leq M$ substencils (Definition 1.2) of equal length of $M - K_s$ intervals, each shifted by 1 cell with respect to its neighbors (K_s is free to take all possible values $\in \{1, \dots, M - 1\}$). We also prove several relations concerning the Lagrange reconstructing polynomial.

In §2 we very briefly summarize those results for the Lagrange reconstructing polynomial and its approximation error obtained in [7] which are necessary in the present work.⁶

In §3 we study the basis polynomials $\alpha_{R_1, M_-, M_+, \ell}(\xi)$ ($\ell \in \{-M_-, \dots, M_+\}$) which [7, Proposition 4.5, p. 287] represent the Lagrange reconstructing polynomial on $S_{i, M_-, M_+} := \{i - M_-, \dots, i + M_+\}$, with coordinates the values $f_{i+\ell} := f(x_i + \ell\Delta x)$ of the cell-averages of the reconstructed function. These results, which include an analysis of the roots of $\alpha_{R_1, M_-, M_+, \ell}(\xi)$ and relations with the polynomials $\lambda_{R_1, M_-, M_+, n}(\xi)$ appearing in the expression of the approximation error of the Lagrange reconstructing polynomial [7, Proposition 4.7, p. 292] are the starting point for the construction of the Lagrange reconstructing polynomial as a combination of the Lagrange reconstructing polynomials on substencils.

In §4 we use the results of §3 to establish a 1-level subdivision rule (Lemma 4.2), by which, applying [12, Lemma 2.1], we construct (Proposition 4.5) an analytical recursive expression of the weight-functions for a general subdivision of an arbitrarily biased stencil on a homogeneous grid. We prove the uniqueness of the rational weight-functions (Proposition 4.7), and we show by studying their poles (all of which are real) that it is always possible to define the weight-functions at half-nodes ($\xi = n + \frac{1}{2}$, $n \in \mathbb{Z}$). Finally, we prove (Theorem 4.14) the convexity of the representation (5a) in the neighborhood of $\xi = \frac{1}{2}$, for all subdivisions (Definition 1.2) for which all of the substencils contain either point i or point $i + 1$ (or both).

2. Reconstruction background

In a recent work [7] we have studied the exact and approximate reconstruction of a function $h(x)$. We have obtained the general analytical solution of the deconvolution of Taylor-series problem [16, (3.13), pp. 244–254], and used this solution in developing analytical relations for the approximation error of polynomial reconstruction on an arbitrary stencil in a homogeneous grid [7]. We briefly summarize those results of [7] which are the starting point of the analysis presented in the present work, and which are necessary for completeness.

Lemma 2.1 (Derivatives of reconstruction pairs). *Let $h = R_{(1;\Delta x)}(f)$ be a reconstruction pair (Definition 1.3), and assume that $f(x)$ and $h(x)$ are of class $C^N[a - \frac{1}{2}\Delta x, b + \frac{1}{2}\Delta x]$. Then*

$$h = R_{(1;\Delta x)}(f) \implies h^{(n)} = R_{(1;\Delta x)}(f^{(n)}) \quad \forall n \in \{1, \dots, N\} \quad (6)$$

PROOF. We have by direct integration

$$\frac{1}{\Delta x} \int_{x-\frac{1}{2}\Delta x}^{x+\frac{1}{2}\Delta x} h'(\zeta) d\zeta = \frac{h(x + \frac{1}{2}\Delta x) - h(x - \frac{1}{2}\Delta x)}{\Delta x} \stackrel{[7, (9)]}{=} f'(x) \quad \forall x \in [a, b] \quad (7)$$

from the fundamental property of reconstruction pairs [7, Lemma 2.2, p. 271], proving (6) for $n = 1$, and by induction $\forall n \in \{1, \dots, N\}$. \square

⁵ In [7, §6.1, pp. 297–300], we had sketched, without giving any proof or analysis, some of the problems which are solved in the present paper. Furthermore, at that time, we had not proven the conjectured convexity [5, 3, 11], in the neighborhood of $\xi = \frac{1}{2}$.

⁶ In the present work we also make extensive use of relations concerning reconstruction pairs [7, Definition 2.1, p. 270], and in particular polynomial reconstruction pairs [7, Theorem 5.1, p. 296].

Expressions for the Lagrange interpolating polynomial and its approximation error are widely available in the literature [8, pp. 186–189], and they are only included in the following to define notation, for completeness, but also to highlight analogies and differences between Lagrange interpolation and Lagrange reconstruction. In [7, Propositions 4.5, 4.6, 4.7] we developed corresponding analytical expressions for the Lagrange reconstructing polynomial, which can be summarized as

Proposition 2.2 (Lagrange polynomial interpolation and reconstruction on S_{i,M_-,M_+} [7]). Assume $M_{\pm} \in \mathbb{Z} : M := M_- + M_+ \geq 0$ (1b). Let $h = R_{(1;\Delta x)}(f)$ be a reconstruction pair (Definition 1.3). Then the Lagrange interpolating (p_{I,M_-,M_+}) and reconstructing (p_{R_1,M_-,M_+}) polynomials of $f(x)$ (Definition 1.4) on S_{i,M_-,M_+} (Definition 1.1) are

$$p_{R_1,M_-,M_+}(x_i + \xi\Delta x; x_i, \Delta x; f) = \sum_{\ell=-M_-}^{M_+} \alpha_{R_1,M_-,M_+, \ell}(\xi) f(x_i + \ell\Delta x) \quad (8a)$$

$$= h(x_i + \xi\Delta x) + \sum_{n=M+1}^{N_{\text{rj}}} \mu_{R_1,M_-,M_+,n}(\xi) \Delta x^n f^{(n)}(x_i) + O(\Delta x^{N_{\text{rj}}+1}) \quad (8b)$$

$$= h(x_i + \xi\Delta x) + \sum_{n=M+1}^{N_{\text{rj}}} \lambda_{R_1,M_-,M_+,n}(\xi) \Delta x^n h^{(n)}(x_i + \xi\Delta x) + O(\Delta x^{N_{\text{rj}}+1}) \quad (8c)$$

$$p_{I,M_-,M_+}(x_i + \xi\Delta x; x_i, \Delta x; f) = \sum_{\ell=-M_-}^{M_+} \alpha_{I,M_-,M_+, \ell}(\xi) f(x_i + \ell\Delta x) \quad (9a)$$

$$= f(x_i + \xi\Delta x) + \sum_{n=M+1}^{N_{\text{rj}}} \mu_{I,M_-,M_+,n}(\xi) \Delta x^n f^{(n)}(x_i) + O(\Delta x^{N_{\text{rj}}+1}) \quad (9b)$$

$$= f(x_i + \xi\Delta x) + \sum_{n=M+1}^{N_{\text{rj}}} \lambda_{I,M_-,M_+,n}(\xi) \Delta x^n f^{(n)}(x_i + \xi\Delta x) + O(\Delta x^{N_{\text{rj}}+1}) \quad (9c)$$

and, provided that $h(x)$ is sufficiently smooth $\forall x \in [x_{i-M_-} - \frac{1}{2}\Delta x, x_{i+M_+} + \frac{1}{2}\Delta x]$, their approximation errors are defined by (8b, 9b), or equivalently by (8c, 9c). The functions $\alpha_{R_1,M_-,M_+, \ell}(\xi)$, $\mu_{R_1,M_-,M_+,n}(\xi)$, $\lambda_{R_1,M_-,M_+,n}(\xi)$, $\alpha_{I,M_-,M_+, \ell}(\xi)$, $\mu_{I,M_-,M_+,n}(\xi)$ and $\lambda_{I,M_-,M_+,n}(\xi)$ are polynomials in

$$\xi := \frac{x - x_i}{\Delta x} \quad (10)$$

with coefficients depending only on M_{\pm} and are defined by

$$\alpha_{R_1,M_-,M_+, \ell}(\xi) := \sum_{m=0}^M \left(\sum_{k=0}^{\lfloor \frac{M-m}{2} \rfloor} \frac{\tau_{2k}(m+2k)!}{m!} ({}_{M_-}^{M_+} V^{-1})_{m+2k+1, \ell+M_-+1} \right) \xi^m \quad ; \quad -M_- \leq \ell \leq +M_+ \quad (11a)$$

$$\mu_{R_1,M_-,M_+,s}(\xi) := \sum_{k=0}^{\lfloor \frac{s}{2} \rfloor} \frac{-\tau_{2k}}{(s-2k)!} \xi^{s-2k} + \sum_{m=0}^M \left(\sum_{k=0}^{\lfloor \frac{M-m}{2} \rfloor} \tau_{2k} V_{M_-,M_+,m+2k,s} \frac{(m+2k)!}{s! m!} \right) \xi^m \quad ; \quad s \geq M+1 \quad (11b)$$

$$\lambda_{R_1,M_-,M_+,n}(\xi) := \sum_{\ell=0}^{n-M-1} \mu_{R_1,M_-,M_+,n-\ell}(\xi) \frac{(-1)^{\ell+1}}{(\ell+1)!} \left((\xi - \frac{1}{2})^{\ell+1} - (\xi + \frac{1}{2})^{\ell+1} \right) \quad ; \quad n \geq M+1 \quad (11c)$$

$$\alpha_{I,M_-,M_+,\ell}(\xi) := \sum_{m=0}^M (M_-^{M_+} V^{-1})_{m+1,\ell+M_-+1} \xi^m \quad ; \quad -M_- \leq \ell \leq +M_+ \quad (12a)$$

$$\mu_{I,M_-,M_+,s}(\xi) := \frac{1}{s!} \left(-\xi^s + \sum_{m=0}^M v_{M_-,M_+,m,s} \xi^m \right) \quad ; \quad s \geq M+1 \quad (12b)$$

$$\lambda_{I,M_-,M_+,n}(\xi) := \sum_{\ell=0}^{n-M-1} \frac{(-\xi)^\ell}{\ell!} \mu_{I,M_-,M_+,n-\ell}(\xi) \quad ; \quad n \geq M+1 \quad (12c)$$

where $(M_-^{M_+} V^{-1})_{ij}$ are the elements of the inverse Vandermonde matrix on S_{i,M_-,M_+} [7, Definition 4.3, p. 283], expressed by [7, (43a,43b), pp. 283–284],⁷ $v_{M_-,M_+,m,s}$ are defined by⁸

$$v_{M_-,M_+,m,k} := \sum_{\ell=-M_-}^{M_+} (M_-^{M_+} V^{-1})_{m+1,\ell+M_-+1} \ell^k \quad ; \quad k \in \mathbb{N}_0 \quad (13)$$

and the numbers τ_n satisfy⁹

$$\tau_0 = 1 \quad ; \quad \tau_{2k} = \sum_{s=0}^{k-1} \frac{-\tau_{2s}}{2^{2k-2s} (2k-2s+1)!} = \sum_{s=1}^k \frac{-\tau_{2k-2s}}{2^{2s} (2s+1)!} \quad k > 0 \quad (14a)$$

$$\tau_{2n+1} = 0 \quad n \geq 0 \quad (14b)$$

□

Remark 2.3 (Alternative expressions for $\alpha_{I,M_-,M_+,\ell}(\xi)$ (12a) and $\alpha_{R_1,M_-,M_+,\ell}(\xi)$ (11a)). The polynomials $\alpha_{I,M_-,M_+,\ell}(\xi)$ are the fundamental functions [8, pp. 183–197] of Lagrange interpolation on the stencil S_{i,M_-,M_+} , and can also be expressed, upon replacing $x_\ell = x_i + \ell\Delta x$ in [8, (9.4), p. 184], as

$$\alpha_{I,M_-,M_+,\ell}(\xi) = \frac{\prod_{\substack{k=-M_- \\ k \neq \ell}}^{M_+} (\xi - k)}{\prod_{\substack{k=-M_- \\ k \neq \ell}}^{M_+} (\ell - k)} \quad (15a)$$

⁷ By [7, Lemma 4.4, pp. 283–284]

$$(M_-^{M_+} V^{-1})_{ij} = \sum_{n=0}^{M+1-i} (M_-)^n \binom{n+i-1}{n} ({}_0^M V^{-1})_{i+n,j} \quad \forall i, j \in \{1, \dots, M+1\} \\ M := M_- + M_+$$

$$({}_0^M V^{-1})_{ij} = (-1)^{i+j} \sum_{k=1}^{M+1} \frac{1}{(k-1)!} \binom{k-1}{j-1} \left[\begin{matrix} k-1 \\ i-1 \end{matrix} \right] \quad \forall i, j \in \{1, \dots, M+1\}$$

⁸ By [7, Lemma 4.4, pp. 283–284]

$$v_{M_-,M_+,m,k} = \sum_{\ell=-M_-}^{M_+} (M_-^{M_+} V^{-1})_{m+1,\ell+M_-+1} \ell^k = \delta_{mk} \quad \begin{matrix} 0 \leq k \leq M \\ 0 \leq m \leq M \end{matrix}$$

$$\sum_{m=0}^M v_{M_-,M_+,m,k} \ell^m = \ell^k \quad \begin{matrix} \forall k \in \mathbb{N}_0 \\ \forall \ell \in \{-M_-, \dots, M_+\} \end{matrix}$$

⁹ By [7, Theorem 2.9, pp. 275–276] the numbers τ_n can be defined as $\tau_n := \frac{1}{n!} g_\tau^{(n)}(0)$ from the generating function $g_\tau(x) := \frac{\frac{1}{2}x}{\sinh \frac{1}{2}x}$

Shu [5, (2.19), p. 336] has shown, using the reconstruction via primitive approach [16, pp. 243–244], that the fundamental functions (11a) of the Lagrange reconstructing polynomial on S_{i,M_-,M_+} , $p_{R_1,M_-,M_+}(x_i + \xi\Delta x; x_i, \Delta x; f)$ (8a), can equivalently be expressed as¹⁰

$$\alpha_{R_1,M_-,M_+, \ell}(\xi) \stackrel{[5, (2.19)]}{=} \sum_{m=\ell+M_-+1}^{M+1} \frac{\sum_{\substack{p=0 \\ p \neq m}}^{M+1} \prod_{\substack{q=0 \\ q \neq m \\ q \neq p}}^{M+1} (\xi - q + \frac{1}{2})}{\prod_{\substack{p=0 \\ p \neq m}}^{M+1} (m - p)} \quad (15b)$$

□

Remark 2.4 (Mapping $R_{(1;\Delta x)}$). By [7, Theorem 5.1, p. 296] the mapping $R_{(1;\Delta x)}$ is a bijection of the $(M+1)$ -dimensional space $\mathbb{R}_M[x]$ of polynomials of degree $\leq M$ onto itself. This implies that polynomial reconstruction pairs are unique [7, Lemma 3.1, p. 296]. Let $p(x) \in \mathbb{R}_M[x]$ be a polynomial of degree $\leq M$. Then by [7, Theorem 5.1, p. 296]

$$\forall p(x) \in \mathbb{R}_M[x] \implies \begin{cases} q(x) := [R_{(1;\Delta x)}(p)](x) \in \mathbb{R}_M[x] \\ \deg(q) = \deg(p) \\ \text{coeff}[x^{\deg(p)}, p(x)] = \text{coeff}[x^{\deg(p)}, q(x)] \end{cases} \quad (16)$$

where the property that the linear operator $R_{(1;\Delta x)} : \mathbb{R}_M[x] \longrightarrow \mathbb{R}_M[x]$ conserves the degree of the polynomial follows directly from [7, Lemma 3.1, p. 277]. Furthermore, the coefficients of the leading power of $p(x) \in \mathbb{R}_M[x]$ and of $q(x) := [R_{(1;\Delta x)}(p)](x) \in \mathbb{R}_M[x]$ can be shown to be equal, by straightforward application of the expression [7, (26f), Lemma 3.1, p. 277].¹¹ One consequence of these properties is that several relations obtained for the interpolating polynomial have their direct analogues for the reconstructing polynomial and *vice-versa*. □

3. Fundamental polynomials of Lagrange interpolation and reconstruction

The construction of a recursive formulation (§4) for the linear weight-functions (5a) is based on the representations of the Lagrange reconstructing polynomial (8a) and of its approximation error (8c). It is therefore necessary to gain some insight on the fundamental functions (11a) of Lagrange reconstruction, and on the truncation-error polynomials (11c).

3.1. Reconstruction pairs of fundamental polynomials

Each of the fundamental polynomials of Lagrange reconstruction $\alpha_{R_1,M_-,M_+, \ell}(\xi)$ (11a) is intimately related to the corresponding fundamental polynomial of Lagrange interpolation $\alpha_{I,M_-,M_+, \ell}(\xi)$ (12a), as can be seen by using (9a, 8a) in the reconstruction-pair-defining relation (3a).

Proposition 3.1 (Reconstruction pairs $\alpha_{R_1,M_-,M_+, \ell} = R_{(1;1)}(\alpha_{I,M_-,M_+, \ell})$). Assume $M_{\pm} \in \mathbb{Z} : M := M_- + M_+ \geq 0$ (1b). The polynomial $\alpha_{R_1,M_-,M_+, \ell}(\xi)$ (11a) appearing in the representation (8a) of the reconstructing polynomial on the stencil S_{i,M_-,M_+} (Definition 1.1) is the reconstruction pair¹² of the corresponding polynomial $\alpha_{I,M_-,M_+, \ell}(\xi)$ (12a) appearing

¹⁰ The correspondence of the present indicial notation with the one used by Shu [5, (2.19), p. 336] is $r_{\text{Shu}} = M_-$, $j_{\text{Shu}} = \ell + M_-$, $k_{\text{Shu}} = M + 1 = M_- + M_+ + 1$, and we use again $x = x_i + \xi\Delta x$ (10).

¹¹ By [7, Lemma 3.1, p. 277] if $\deg(p) = M$, and $p(x_i + \xi\Delta x) = \sum_{m=0}^M c_{pm} \xi^m$ then $q(x_i + \xi\Delta x) := [R_{(1;\Delta x)}(p)](x_i + \xi\Delta x) = \sum_{m=0}^M c_{qm} \xi^m$ and by [7, (26f), p. 277]

$$c_{qM} \stackrel{[7, (26f)]}{=} \frac{1}{M!} \sum_{k=0}^{\lfloor \frac{M-M}{2} \rfloor} \tau_{2k} c_{pM+2k} (M+2k)! \stackrel{(14a)}{=} c_{pM}$$

¹² on a unit-spacing grid, $\Delta x = 1$

in the representation (9a) of the interpolating polynomial on the same stencil

$$\alpha_{R_1, M_-, M_+, \ell}(\xi) = [R_{(1;1)}(\alpha_{I, M_-, M_+, \ell})](\xi) \iff \alpha_{I, M_-, M_+, \ell}(\xi) = \int_{\xi - \frac{1}{2}}^{\xi + \frac{1}{2}} \alpha_{R_1, M_-, M_+, \ell}(\eta) d\eta \quad \begin{cases} \forall \ell \in \{-M_-, \dots, M_+\} \\ \forall \xi \in \mathbb{R} \end{cases} \quad (17)$$

PROOF. By Definition 1.4 of the reconstructing polynomial, we have, using (3a)

$$p_{I, M_-, M_+}(x_i + \xi \Delta x; x_i, \Delta x; f) = \frac{1}{\Delta x} \int_{x_i + \xi \Delta x - \frac{1}{2} \Delta x}^{x_i + \xi \Delta x + \frac{1}{2} \Delta x} p_{R_1, M_-, M_+}(\zeta; x_i, \Delta x; f) d\zeta \quad (18a)$$

and using the representation (8a) of the reconstructing polynomial and the representation (9a) of the interpolating polynomial, we readily obtain by (18a)

$$\sum_{\ell=-M_-}^{M_+} \left(\alpha_{I, M_-, M_+, \ell}(\xi) - \int_{\xi - \frac{1}{2}}^{\xi + \frac{1}{2}} \alpha_{R_1, M_-, M_+, \ell}(\eta) d\eta \right) f(x_i + \ell \Delta x) \stackrel{(8a, 9a, 18a)}{=} 0 \quad \begin{cases} \forall \xi \in \mathbb{R} \\ \forall x_i \in \mathbb{R} \\ \forall \Delta x \in \mathbb{R}_{>0} \\ \forall f : \mathbb{R} \rightarrow \mathbb{R} \end{cases} \quad (18b)$$

Since (18b) is valid $\forall f : \mathbb{R} \rightarrow \mathbb{R}$, it proves, using the definitions (3a, 3b), (17). \square

Lemma 3.2 ($\alpha_{R_1, M_-, M_+, \ell}(\xi) \neq 0_{\mathbb{R}_M[\xi]} \neq \alpha_{I, M_-, M_+, \ell}(\xi)$). Assume $M_{\pm} \in \mathbb{Z} : M := M_- + M_+ \geq 0$ (1b) and let

$$0_{\mathbb{R}_M[\xi]}(\xi) := \sum_{m=0}^M 0 \xi^m = 0 \quad (19a)$$

denote the 0-element of the space $\mathbb{R}_M[\xi]$ of all polynomials of degree $\leq M$. None of the polynomials (11a, 12a), of degree M in ξ is identically 0

$$\alpha_{R_1, M_-, M_+, \ell}(\xi) \neq 0_{\mathbb{R}_M[\xi]}(\xi) \neq \alpha_{I, M_-, M_+, \ell}(\xi) \quad \forall \ell \in \{-M_-, \dots, M_+\} \quad (19b)$$

Furthermore

$$\deg[\alpha_{R_1, M_-, M_+, \ell}(\xi)] = \deg[\alpha_{I, M_-, M_+, \ell}(\xi)] = M \quad (19c)$$

$$\text{coeff}[\xi^M, \alpha_{R_1, M_-, M_+, \ell}(\xi)] = \text{coeff}[\xi^M, \alpha_{I, M_-, M_+, \ell}(\xi)] = (-1)^{\ell+M_+} \frac{1}{M!} \binom{M}{\ell + M_-} \neq 0 \quad \forall \ell \in \{-M_-, \dots, M_+\} \quad (19d)$$

PROOF. It is well known, and also obvious from the expression (15a), that the fundamental polynomials $\alpha_{I, M_-, M_+, \ell}(\xi)$ of Lagrange interpolation on a stencil of $M+1$ equidistant points are $\neq 0_{\mathbb{R}_M[\xi]}(\xi)$. Since by (17) $\alpha_{I, M_-, M_+, \ell}(\xi) \neq 0_{\mathbb{R}_M[\xi]}(\xi)$ is equal to the definite integral of $\alpha_{R_1, M_-, M_+, \ell}(\xi)$ in the interval $[\xi - \frac{1}{2}, \xi + \frac{1}{2}]$, it follows that $\alpha_{R_1, M_-, M_+, \ell}(\xi) \neq 0_{\mathbb{R}_M[\xi]}(\xi)$. By (11a, 12a) we have (19c). It is easy to show by direct computation¹³ that $\text{coeff}[\xi^M, \alpha_{R_1, M_-, M_+, \ell}(\xi)]$ is given by

¹³ By Proposition 2.2

$$\begin{aligned} \text{coeff}[\xi^M, \alpha_{R_1, M_-, M_+, \ell}(\xi)] &\stackrel{(11a)}{=} \sum_{k=0}^{\lfloor \frac{M-M}{2} \rfloor} \frac{\tau_{2k}(M+2k)!}{M!} (M_+ V^{-1})_{M+2k+1, \ell+M_-+1} = \frac{\tau_0 M!}{M!} (M_+ V^{-1})_{M+1, \ell+M_-+1} \stackrel{(14a)}{=} (M_+ V^{-1})_{M+1, \ell+M_-+1} \\ &\stackrel{[7, (43a)]}{=} \sum_{n=0}^{M+1-M-1} (M_-)^n \binom{n+M+1-1}{n} (M_0 V^{-1})_{M+1+n, \ell+M_-+1} = (M_-)^0 \binom{M}{0} (M_0 V^{-1})_{M+1, \ell+M_-+1} \\ &= (M_0 V^{-1})_{M+1, \ell+M_-+1} \stackrel{[7, (43b)]}{=} (-1)^{M+1+\ell+M_-+1} \sum_{k=1}^{M+1} \frac{1}{(k-1)!} \binom{k-1}{\ell+M_-+1-1} \left[\begin{matrix} k-1 \\ M+1-1 \end{matrix} \right] \\ &= (-1)^{M+\ell+M_-} \frac{1}{M!} \binom{M}{\ell+M_-} \left[\begin{matrix} M \\ M \end{matrix} \right] = (-1)^{M+\ell+M_-} \frac{1}{M!} \binom{M}{\ell+M_-} \\ &\stackrel{M:=M_-+M_+}{=} (-1)^{M_++\ell+2M_-} \frac{1}{M!} \binom{M}{\ell+M_-} = (-1)^{\ell+M_+} \frac{1}{M!} \binom{M}{\ell+M_-} \neq 0 \quad \forall \ell \in \{-M_-, \dots, M_+\} \stackrel{(1b)}{\implies} \ell+M_- \leq M \end{aligned}$$

where we used the expressions [7, (43a, 43b), pp. 283–284] for the elements of the inverse of the Vandermonde matrix (fn7, p. 7), and well known properties of the unsigned Stirling numbers of the first kind [17, Tab. 264, p. 264], $m < n \neq 0 \implies \left[\begin{matrix} m \\ n \end{matrix} \right] = 0 \forall m, n \in \mathbb{N}$ and $\left[\begin{matrix} n \\ n \end{matrix} \right] = 1 \forall n \in \mathbb{N}_0$.

(19d). By (16) this is also the coefficient of ξ^M of the polynomial $\alpha_{I,M_-,M_+, \ell}(\xi)$ (12a), which (Proposition 3.1) is the reconstruction pair of $\alpha_{R_1,M_-,M_+, \ell}(\xi)$, proving (19d). \square

Proposition 3.3 (Basis $\{\alpha_{R_1,M_-,M_+, \ell}(\xi), \ell \in \{-M_-, \dots, M_+\}\}$). Assume $M_{\pm} \in \mathbb{Z} : M := M_- + M_+ \geq 0$ (1b). The $(M+1)$ unique polynomials $\alpha_{R_1,M_-,M_+, \ell}(\xi) \neq 0_{\mathbb{R}_M[\xi]}$ (11a), $\ell \in \{-M_-, \dots, M_+\}$, constitute a basis of the $(M+1)$ -dimensional space $\mathbb{R}_M[\xi]$ of all polynomials of degree $\leq M$.

PROOF. This can be proved either by Proposition 3.1 or directly.

Proof by Proposition 3.1: It is a well-known fact [8], and also obvious from (15a), that the $(M+1)$ unique polynomials $\{\alpha_{I,M_-,M_+, \ell}(\xi), \ell \in \{-M_-, \dots, M_+\}\}$ are linearly independent and span the $(M+1)$ -dimensional space $\mathbb{R}_M[\xi]$ of all polynomials of degree $\leq M$ in ξ . By Lemma 3.1 each polynomial $\alpha_{R_1,M_-,M_+, \ell} = R_{(1;\Delta x)}(\alpha_{I,M_-,M_+, \ell})$ (17), and by [7, Theorem 5.1, p. 296] the mapping $R_{(1;\Delta x)} : \mathbb{R}_M[\xi] \rightarrow \mathbb{R}_M[\xi]$ is a bijection. Hence the image of $\{\alpha_{I,M_-,M_+, \ell}(\xi), \ell \in \{-M_-, \dots, M_+\}\}$, $\{\alpha_{R_1,M_-,M_+, \ell}(\xi), \ell \in \{-M_-, \dots, M_+\}\}$ is also a basis of $\mathbb{R}_M[\xi]$.

Direct proof: Existence of the polynomials $\alpha_{R_1,M_-,M_+, \ell}(\xi)$ satisfying (8a) was proved by construction [7, Proposition 4.5, p. 287] yielding (11a). Recall that by [7, Theorem 5.1, p. 296] the linear operator $R_{(1;\Delta x)}$ (Definition 1.3) is a bijection of the vector space $\mathbb{R}_M[x]$ of all polynomials of degree $\leq M$ onto itself. Obviously, by [7, Lemma 3.1, p. 277] the same properties apply to the inverse operator $R_{(1;\Delta x)}^{-1}$. Since, $\forall p(x) \in \mathbb{R}_M[x] \implies p^{(s)}(x) = 0 \forall s \geq M+1$, the reconstructing polynomial (Definition 1.4) of $p(x)$ on the stencil $S_{i,M_-,M_+} := \{i-M_-, \dots, i+M_+\}$ (Definition 1.1) is exactly equal to the reconstruction pair of $p(x)$ (Definition 1.4), $q(x) := [R_{(1;\Delta x)}(p)](x)$, by (8c). By (8a) we have

$$q(x) \stackrel{(8a, 8c)}{=} \sum_{\ell=-M_-}^{M_+} \alpha_{R_1,M_-,M_+, \ell} \left(\frac{x-x_i}{\Delta x} \right) [R_{(1;\Delta x)}^{-1}(q)](x_i + \ell \Delta x) \quad \begin{cases} \forall q(x) \in \mathbb{R}_M[x] \\ \forall x \in \mathbb{R} \\ \forall x_i \in \mathbb{R} \\ \forall \Delta x \in \mathbb{R}_{>0} \end{cases} \quad (20a)$$

Since (20a) holds $\forall x_i \in \mathbb{R}$ and $\forall \Delta x \in \mathbb{R}_{>0}$ we may set $x_i = 0$ and $\Delta x = 1$ in (20a) to obtain

$$q(x) = \sum_{\ell=-M_-}^{M_+} \alpha_{R_1,M_-,M_+, \ell}(x) [R_{(1;1)}^{-1}(q)](\ell) \quad \begin{cases} \forall q(x) \in \mathbb{R}_M[x] \\ \forall x \in \mathbb{R} \end{cases} \quad (20b)$$

By (20b), the $M+1$ polynomials $\alpha_{R_1,M_-,M_+, \ell}(x), \ell \in \{-M_-, \dots, M_+\}$ span $\mathbb{R}_M[x]$, and since $\dim(\mathbb{R}_M[x]) = M+1$ they form a basis of $\mathbb{R}_M[x]$. They are therefore linearly independent [18], and as a consequence $\neq 0_{\mathbb{R}_M[\xi]}$ (19b), a fact already proven in Lemma 3.2. \square

3.2. Roots of fundamental polynomials

Because of (17) for every value returned by the polynomial $\alpha_{I,M_-,M_+, \ell}(\xi_I)$ at point $\xi_I \in \mathbb{R}$, there exists a nearby point $\xi_{R_1} \in \mathbb{R}$ such that $\alpha_{R_1,M_-,M_+, \ell}(\xi_{R_1}) = \alpha_{I,M_-,M_+, \ell}(\xi_I)$, the distance between the 2 points being $|\xi_{R_1} - \xi_I| < \frac{1}{2}$. This can be formalized as

Lemma 3.4 $(\alpha_{I,M_-,M_+, \ell}([\xi_1, \xi_2]) \subseteq \alpha_{R_1,M_-,M_+, \ell}((\xi_1 - \frac{1}{2}, \xi_2 + \frac{1}{2})) \forall \xi_1, \xi_2 \in \mathbb{R} : \xi_1 \leq \xi_2)$. Assume $M_{\pm} \in \mathbb{Z} : M := M_- + M_+ \geq 0$ (1b). Then

$$\forall \ell \in \{-M_-, \dots, M_+\} \quad \forall \xi_I \in \mathbb{R} \quad \exists \xi_{R_1} \in (\xi_I - \frac{1}{2}, \xi_I + \frac{1}{2}) \subset \mathbb{R} : \alpha_{R_1,M_-,M_+, \ell}(\xi_{R_1}) = \alpha_{I,M_-,M_+, \ell}(\xi_I) \quad (21a)$$

where $\alpha_{R_1,M_-,M_+, \ell}(\xi)$ (11a) and $\alpha_{I,M_-,M_+, \ell}(\xi)$ (12a) are the fundamental polynomials of Lagrange reconstruction and interpolation, respectively (Proposition 2.2), implying that

$$\forall \ell \in \{-M_-, \dots, M_+\} \quad \alpha_{I,M_-,M_+, \ell}([\xi_1, \xi_2]) \subseteq \alpha_{R_1,M_-,M_+, \ell}((\xi_1 - \frac{1}{2}, \xi_2 + \frac{1}{2})) \forall \xi_1, \xi_2 \in \mathbb{R} : \xi_1 \leq \xi_2 \quad (21b)$$

PROOF. The proof follows immediately from Proposition 3.1. By (17)

$$\forall \xi_I \in \mathbb{R} \quad \alpha_{I,M_-,M_+, \ell}(\xi_I) = \int_{\xi_I - \frac{1}{2}}^{\xi_I + \frac{1}{2}} \alpha_{R_1,M_-,M_+, \ell}(\eta) d\eta \quad \forall \ell \in \{-M_-, \dots, M_+\} \quad (22)$$

Using the mean value theorem for the definite integral [19, p. 352] in (22) yields (21a), from which (21b) is easily proved by contradiction. \square

The fundamental polynomials of the Lagrange interpolating polynomial $\alpha_{I,M_-,M_+,\ell}(\xi)$ (12a) are polynomials of degree M in ξ (12a), and it is well known [8] and obvious from their expression (15a) that their M roots are the integer nodes $\{-M_-, \dots, M_+\} \setminus \{\ell\}$

$$\alpha_{I,M_-,M_+,\ell}(n) = 0 \quad \forall n \in \{-M_-, \dots, M_+\} \setminus \{\ell\} \quad \forall \ell \in \{-M_-, \dots, M_+\} \quad (23a)$$

$$\alpha_{I,M_-,M_+,\ell}(\xi) \neq 0 \quad \forall \xi \in \mathbb{R} \setminus \{-M_-, \dots, M_+\} \setminus \{\ell\} \quad \forall \ell \in \{-M_-, \dots, M_+\} \quad (23b)$$

The fundamental polynomials of the Lagrange reconstructing polynomial $\alpha_{R_1,M_-,M_+,\ell}(\xi)$ (11a, 15b) are also polynomials of degree M in ξ (11a), but the expressions (11a, 15b) are too complicated to directly give information about their roots. It is nonetheless easy, using Lemma 3.4, to show that

Proposition 3.5 (Roots of $\alpha_{R_1,M_-,M_+,\ell}(\xi)$ (11a)). *Assume $M_{\pm} \in \mathbb{Z} : M := M_- + M_+ \geq 0$ (1b). The M roots of the degree M in ξ polynomials $\alpha_{R_1,M_-,M_+,\ell}(\xi)$ (11a) are all distinct and real, and there is exactly 1 root in each open interval $(n - \frac{1}{2}, n + \frac{1}{2}) \forall n \in \{-M_-, \dots, M_+\} \setminus \{\ell\}$, ie*

$$\left. \begin{array}{l} \forall n \in \{-M_-, \dots, M_+\} \setminus \{\ell\} \\ \forall \ell \in \{-M_-, \dots, M_+\} \end{array} \right\} \exists! \xi_{R_1,M_-,M_+,\ell,n} \in (n - \frac{1}{2}, n + \frac{1}{2}) \subset \mathbb{R} : \alpha_{R_1,M_-,M_+,\ell}(\xi_{R_1,M_-,M_+,\ell,n}) = 0 \quad (24a)$$

$$\alpha_{R_1,M_-,M_+,\ell}(\xi) \neq 0 \quad \forall \xi \in \mathbb{R} \setminus \left\{ \xi_{R_1,M_-,M_+,\ell,n} : n \in \{-M_-, \dots, M_+\} \setminus \{\ell\} \right\} \quad \forall \ell \in \{-M_-, \dots, M_+\} \quad (24b)$$

PROOF. The proof follows immediately from Lemma 3.4, by writing (21a) at each of the M roots (23a) of $\alpha_{I,M_-,M_+,\ell}(\xi)$. By Proposition 3.2 the polynomial $\alpha_{R_1,M_-,M_+,\ell}(\xi) \neq 0_{\mathbb{R}_M[\xi]}(\xi)$. Furthermore $\deg[\alpha_{R_1,M_-,M_+,\ell}(\xi)] = M$ (19c), and since there are exactly $M := M_- + M_+$ elements in $\{-M_-, \dots, M_+\} \setminus \{\ell\}$, the roots (24a) are, by the fundamental theorem of algebra and its corollaries [19, pp. 282–289], the only roots of $\alpha_{R_1,M_-,M_+,\ell}(\xi)$, which proves (24b), and uniqueness ($\exists!$) in (24a), by contradiction. \square

Remark 3.6 (Extrema of $\alpha_{R_1,M_-,M_+,\ell}(\xi)$ (11a)). It is straightforward to show that each fundamental polynomial of Lagrange reconstruction $\alpha_{R_1,M_-,M_+,\ell}(\xi)$ (11a) has $M - 1$ extrema, where $\alpha'_{R_1,M_-,M_+,\ell}(\xi) = 0$, one in each interval between 2 consecutive roots $\xi_{R_1,M_-,M_+,\ell,n}$ (24a). Indeed, for any nonzero polynomial $p(\xi) \in \mathbb{R}_M[\xi]$ with M distinct real roots we know, by Rolle's theorem [19, pp. 215–216], that there is a point where $p'(\xi) = 0$ in each of the $M - 1$ intervals between 2 consecutive distinct real roots, these $M - 1$ points being exactly the $M - 1$ roots of $p'(\xi) \in \mathbb{R}_{M-1}[\xi]$. Both $\alpha_{R_1,M_-,M_+,\ell}(\xi) \in \mathbb{R}_M[\xi]$ (11a), by Proposition 3.5, and $\alpha_{I,M_-,M_+,\ell}(\xi) \in \mathbb{R}_M[\xi]$ (12a), by (23), have M real distinct roots. Therefore, $\alpha'_{R_1,M_-,M_+,\ell}(\xi) \in \mathbb{R}_{M-1}[\xi]$ and $\alpha'_{I,M_-,M_+,\ell}(\xi) \in \mathbb{R}_{M-1}[\xi]$ have $M - 1$ real distinct roots, corresponding to the $M - 1$ extrema of $\alpha_{R_1,M_-,M_+,\ell}(\xi)$ and $\alpha_{I,M_-,M_+,\ell}(\xi)$, respectively. Since (Proposition 3.1) $\alpha_{R_1,M_-,M_+,\ell}(\xi) = [R_{(1;1)}(\alpha_{I,M_-,M_+,\ell})](\xi)$, by Lemma 2.1, $\alpha'_{R_1,M_-,M_+,\ell}(\xi) = [R_{(1;1)}(\alpha'_{I,M_-,M_+,\ell})](\xi)$, so that their corresponding $M - 1$ distinct real roots, which are also the corresponding extrema of $\alpha_{R_1,M_-,M_+,\ell}(\xi) = [R_{(1;1)}(\alpha_{I,M_-,M_+,\ell})](\xi)$, are distant by $< \frac{1}{2}$ (Lemma 3.4). \square

Proposition 3.7 (Factorization of $\alpha_{I,M_-,M_+,\ell}(\xi)$ (12a) and $\alpha_{R_1,M_-,M_+,\ell}(\xi)$ (11a)). *Assume $M_{\pm} \in \mathbb{Z} : M := M_- + M_+ \geq 0$, and $\ell \in \{-M_-, \dots, M_+\}$. Then the fundamental polynomials of Lagrange reconstruction $\alpha_{R_1,M_-,M_+,\ell}(\xi)$ (11a) and interpolation $\alpha_{I,M_-,M_+,\ell}(\xi)$ (12a) on the stencil S_{i,M_-,M_+} (Definition 1.1) can be factorized as*

$$\alpha_{R_1,M_-,M_+,\ell}(\xi) = (-1)^{\ell+M_+} \frac{1}{M!} \binom{M}{\ell + M_-} \prod_{\substack{n=-M_- \\ n \neq \ell}}^{M_+} (\xi - \xi_{R_1,M_-,M_+,\ell,n}) \quad (25a)$$

$$\alpha_{I,M_-,M_+,\ell}(\xi) = (-1)^{\ell+M_+} \frac{1}{M!} \binom{M}{\ell + M_-} \prod_{\substack{n=-M_- \\ n \neq \ell}}^{M_+} (\xi - n) \quad (25b)$$

where $\xi_{R_1,M_-,M_+,\ell,n} \in \mathbb{R}$ ($n \in \{-M_-, \dots, M_+\} \setminus \{\ell\}$) are the M real and distinct roots of $\alpha_{R_1,M_-,M_+,\ell}(\xi)$ (Proposition 3.5).

PROOF. Every polynomial $p(x) \in \mathbb{R}[x]$ can be factorized as $p(x) = \text{coeff}[x^{\deg(p)}, p(x)] \prod_{n=1}^{\deg(p)} (x - x_{p_n})$, where $x_{p_n} \in \mathbb{C}$ ($n \in \{1, \dots, \deg(p)\}$) are its $\deg(p) \in \mathbb{N}$ roots [19, pp. 284–285]. We know that $\deg(\alpha_{R_1, M_-, M_+, \ell}) = \deg(\alpha_{I, M_-, M_+, \ell}) = M_- + M_+ = M$ (19c). The M roots of $\alpha_{I, M_-, M_+, \ell}(\xi)$ are $n \in \{-M_-, \dots, M_+\} \setminus \{\ell\} \subset \mathbb{Z}$ (23), and the M roots of $\alpha_{R_1, M_-, M_+, \ell}(\xi)$, $\xi_{R_1, M_-, M_+, \ell, n} \in \mathbb{R}$ ($n \in \{-M_-, \dots, M_+\} \setminus \{\ell\}$) are real (Proposition 3.5). Furthermore $\text{coeff}[\xi^M, \alpha_{R_1, M_-, M_+, \ell}(\xi)] = \text{coeff}[\xi^M, \alpha_{I, M_-, M_+, \ell}(\xi)]$ are given by (19d). These facts prove (25).¹⁴ \square

Example 3.8 (Fundamental polynomials $\alpha_{I, M_-, M_+, \ell}(\xi)$ (12a) and $\alpha_{R_1, M_-, M_+, \ell}(\xi)$ (11a)). Consider the fundamental polynomials of Lagrange reconstruction, $\alpha_{R_1, M_-, M_+, \ell}(\xi)$ (11a), and the corresponding fundamental polynomials of Lagrange interpolation, $\alpha_{I, M_-, M_+, \ell}(\xi)$ (12a), on the stencils (Definition 1.1) $S_{i,3,3}$ (Fig. 1) and $S_{i,3,4}$ (Fig. 2). We know that the corresponding polynomials, $\alpha_{R_1, M_-, M_+, \ell}(\xi) \in \mathbb{R}_M[\xi]$ (11a) and $\alpha_{I, M_-, M_+, \ell}(\xi) \in \mathbb{R}_M[\xi]$ (12a), $\forall \ell \in \{-M_-, \dots, M_+\}$, have $M := M_- + M_+$ distinct real roots (Proposition 3.5), each root of the fundamental polynomial of Lagrange reconstruction $\xi_{R_1, M_-, M_+, \ell, n}$ (24a) being close to the root $n \in \{-M_-, \dots, M_+\} \setminus \{\ell\}$ of the corresponding fundamental polynomial of Lagrange interpolation (23a), viz $|\xi_{R_1, M_-, M_+, \ell, n} - n| < \frac{1}{2}$ (24a). Furthermore (Remark 3.6) both $\alpha_{I, M_-, M_+, \ell}(\xi)$ (12a) and $\alpha_{R_1, M_-, M_+, \ell}(\xi)$ (11a) have $M - 1$ corresponding extrema, again distant $< \frac{1}{2}$. For these reasons the shapes of $\alpha_{I, M_-, M_+, \ell}(\xi)$ (12a) and $\alpha_{R_1, M_-, M_+, \ell}(\xi)$ (11a) are quite similar (Figs. 1, 2). For the stencil $S_{i,3,3}$ (Fig. 1) which is symmetric around $\xi = 0$, we observe that $\xi_{R_1,3,3,\ell,n} \notin \mathbb{Z} \forall \ell \in \{-3, +3\} \forall n \in \{-3, \dots, +3\} \setminus \{\ell\}$. On the contrary, for the stencil $S_{i,3,4}$ (Fig. 2) which is symmetric around $\xi = \frac{1}{2}$, we observe that there are two integer roots, $\xi_{R_1,3,4,-3,1} = +1 \in \mathbb{Z}$ and $\xi_{R_1,3,4,+4,0} = 0 \in \mathbb{Z}$. Although we have not worked out a formal proof concerning integer roots, we can formulate the following conjecture (Result 3.9), obtained using symbolic computation. \square

Result 3.9 (Integer roots of $\alpha_{R_1, M_-, M_+, \ell}(\xi)$ (11a)). *Let*

$$M_{\pm} \in \{-20, \dots, +20\} : M := M_- + M_+ \geq 1 \quad (26a)$$

Then for M even

$$M = 2k ; k \in \mathbb{N} \implies \xi_{R_1, M_-, M_+, \ell, n} \notin \mathbb{Z} \quad \begin{cases} \forall \ell \in \{-M_-, \dots, M_+\} \\ \forall n \in \{-M_-, \dots, M_+\} \setminus \{\ell\} \end{cases} \quad (26b)$$

and for M odd

$$M = 2k + 1 ; k \in \mathbb{N}_0 \implies \begin{cases} \xi_{R_1, M_-, M_+, -M_-, -M_- + \lceil \frac{M}{2} \rceil} = -M_- + \lceil \frac{M}{2} \rceil = \frac{M_+ - M_- + 1}{2} \\ \xi_{R_1, M_-, M_+, +M_+, +M_+ - \lceil \frac{M}{2} \rceil} = +M_+ - \lceil \frac{M}{2} \rceil = \frac{M_+ - M_- - 1}{2} \\ \xi_{R_1, M_-, M_+, \ell, n} \notin \mathbb{Z} \quad \begin{cases} \forall \ell \in \{-M_- + 1, \dots, M_+ - 1\} \\ \forall n \in \{-M_-, \dots, M_+\} \setminus \{\ell\} \end{cases} \end{cases} \quad (26c)$$

VERIFICATION. By Proposition 3.5, we know that all of the roots of the basis polynomials $\alpha_{R_1, M_-, M_+, \ell}(\xi)$ are real. Since by (11a) $\deg(\alpha_{R_1, M_-, M_+, \ell}) = M$ ($M := M_- + M_+$), there are M real roots, with exactly 1 root in each of the M open intervals $(n - \frac{1}{2}, n + \frac{1}{2}) \forall n \in \{-M_-, \dots, M_+\} \setminus \{\ell\}$ (24a). Hence, if $\alpha_{R_1, M_-, M_+, \ell}(\xi)$ has integer roots, these must belong to the set $\{-M_-, \dots, M_+\} \setminus \{\ell\} \subset \mathbb{Z}$, ie if $\alpha_{R_1, M_-, M_+, \ell}(\xi)$ has integer roots these must lie on the points of the stencil $S_{i, M_-, M_+} := \{i - M_-, \dots, i + M_+\}$ (1c), except the point $i + \ell$ itself. As a consequence, the result was obtained by direct calculation of $\alpha_{R_1, M_-, M_+, \ell}(\xi)$ (11a) $\forall \ell, n \in \{-M_-, \dots, M_+\}$ for the range of stencils studied. \square

¹⁴ Notice that by comparison of (25b) with (15a)

$$\frac{1}{\prod_{\substack{k=-M_- \\ k \neq \ell}}^{M_+} (\ell - k)} = (-1)^{\ell + M_+} \frac{1}{M!} \binom{M}{\ell + M_-}$$

as can be easily verified by direct computation.

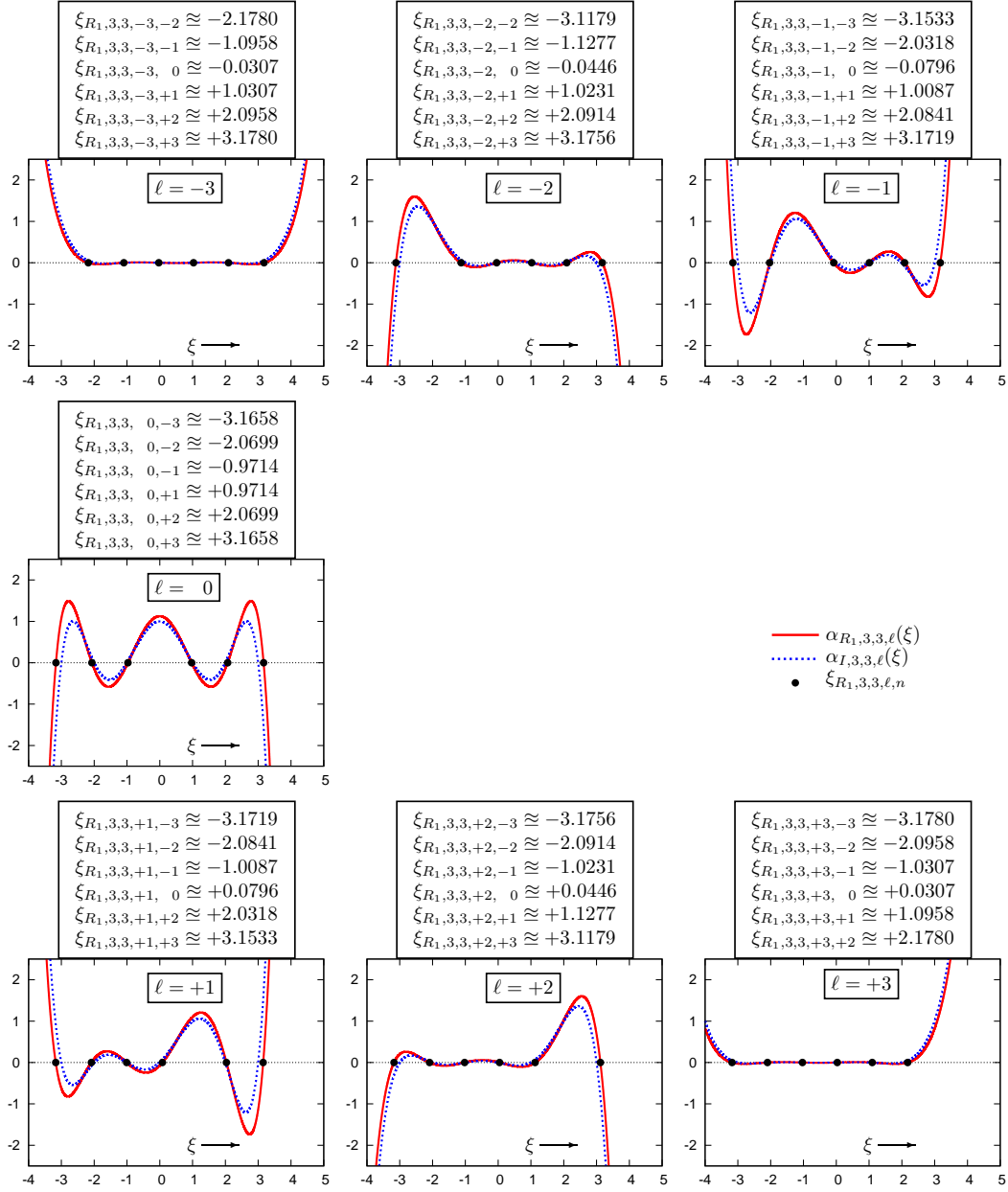


Figure 1: Fundamental polynomials (Proposition 2.2) of Lagrange interpolation, $\alpha_{I,M_-,M_+,\ell}(\xi)$ (12a), and reconstruction, $\alpha_{R1,M_-,M_+,\ell}(\xi)$ (11a), on the stencil $S_{i,3,3}$ ($\ell \in \{-3, \dots, +3\}$), and locations of the 6 real roots of each $\alpha_{R1,3,3,\ell}(\xi) \in \mathbb{R}_6[\xi]$ (Proposition 3.5), $\xi_{R1,M_-,M_+,\ell,n}$ ($n \in \{-M_-, \dots, M_+\} \setminus \{\ell\}$), appearing in the factorization (25a) of $\alpha_{R1,M_-,M_+,\ell}(\xi)$ (Proposition 3.7); notice that in the present case ($M := M_- + M_+ = 6$) $\xi_{R1,3,3,\ell,n} \notin \mathbb{Z} \forall \ell \in \{-3, +3\} \forall n \in \{-3, \dots, +3\} \setminus \{\ell\}$ (Result 3.9).

3.3. Some identities concerning the fundamental polynomials of Lagrange reconstruction

To build (§4) the recursive construction of the weight-functions $[\sigma_{R1,M_-,M_+,K_s,k_s}(\xi)]$ for the combination (5a) of the polynomial reconstructions $p_{R1,M_-,M_+,K_s,k_s}(x_i + \xi\Delta x; x_i, \Delta x; f)$ on the substencils (Definition 1.2) of S_{i,M_-,M_+} (Definition 1.1) to the polynomial reconstruction $p_{R1,M_-,M_+}(x_i + \xi\Delta x; x_i, \Delta x; f)$ on the big stencil S_{i,M_-,M_+} , we will first examine (Lemma 4.2) the elementary subdivision of S_{i,M_-,M_+} (Definition 1.1) into the substencils S_{i,M_-,M_+} (which omits the leftmost point $i - M_-$) and S_{i,M_-,M_+} (which omits the rightmost point $i + M_+$). To obtain the general result for $\sigma_{R1,M_-,M_+,1,0}(\xi)$ and $\sigma_{R1,M_-,M_+,1,1}(\xi)$ (Lemma 4.2) we need to show that the leading terms of the approximation error (8c) of the Lagrange reconstructing polynomial, on 2 overlapping stencils of equal length, but shifted by 1 cell,

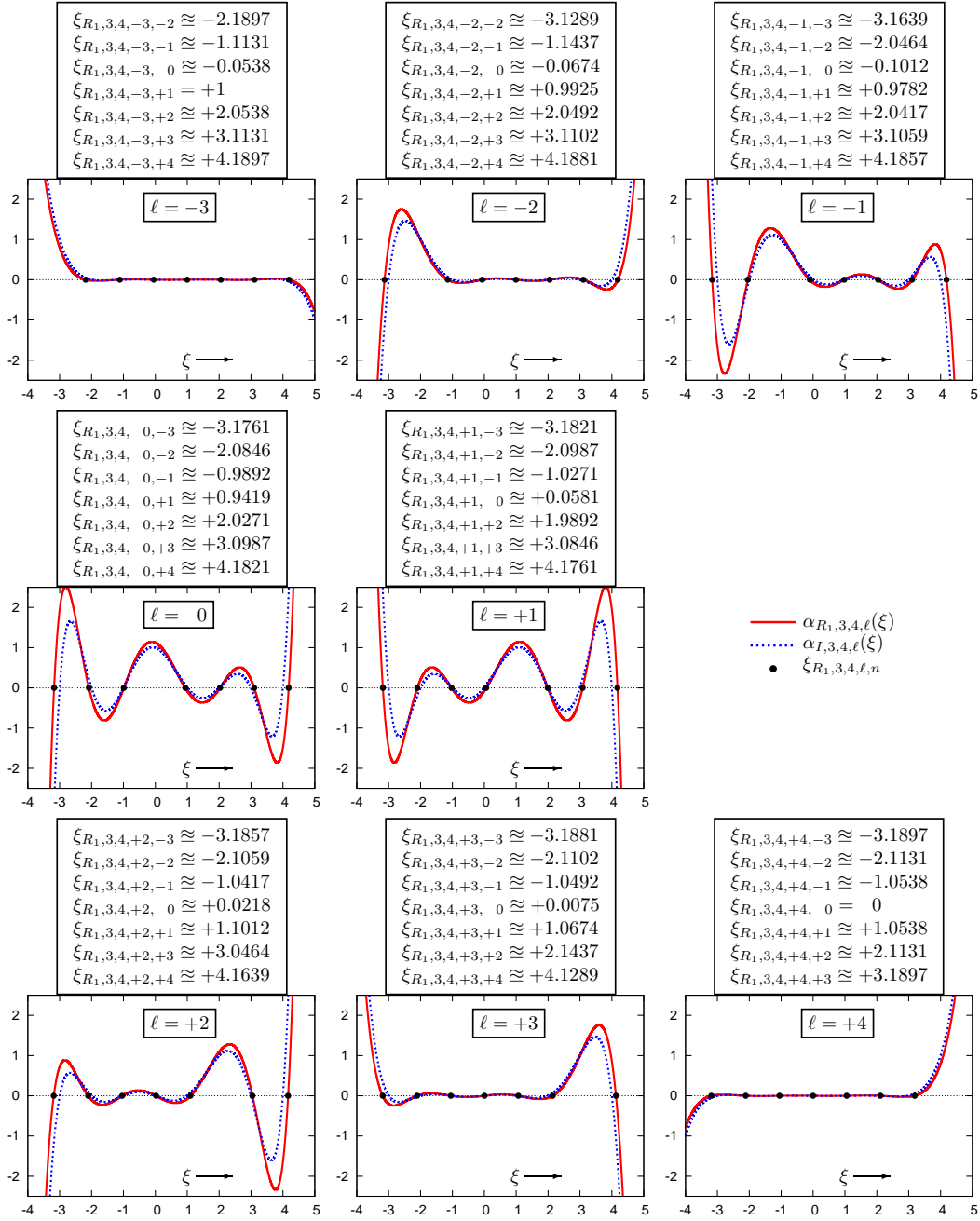


Figure 2: Fundamentat polynomials (Proposition 2.2) of Lagrange interpolation, $\alpha_{I,M_-,M_+,l}(\xi)$ (12a), and reconstruction, $\alpha_{R1,M_-,M_+,l}(\xi)$ (11a), on the stencil $s_{i,3,4}$ ($\ell \in \{-3, \dots, +4\}$), and locations of the 7 real roots of each $\alpha_{R1,3,4,\ell}(\xi) \in \mathbb{R}_7[\xi]$ (Proposition 3.5), $\xi_{R1,M_-,M_+,l,n}$ ($n \in \{-M_-, \dots, M_+\} \setminus \{\ell\}$), appearing in the factorization (25a) of $\alpha_{R1,M_-,M_+,l}(\xi)$ (Proposition 3.7); notice that in the present case ($M := M_- + M_+ = 7$) $\xi_{R1,3,4,-3,1} = +1 \in \mathbb{Z}$ and $\xi_{R1,3,4,+4,0} = 0 \in \mathbb{Z}$ (Result 3.9).

are different. Since the error-expansion (8c) polynomials (11c), $\lambda_{R1,M_-,M_+,M}(\xi)$ and $\lambda_{R1,M_-,M_+,M-1}(\xi)$, are of degree M [7, fn8, p. 294, Proposition 4.7], they can be projected on the basis $\{\alpha_{R1,M_-,M_+,l}(\xi), \ell \in \{-M_-, \dots, M_+\}\}$ of $\mathbb{R}_M[\xi]$ (Proposition 3.3), and the same projection is possible for the polynomials $\{\alpha_{R1,M_-,M_+,l}(\xi), \ell \in \{-M_- + 1, \dots, M_+\}\} \in \mathbb{R}_{M-1}[\xi] \subset \mathbb{R}_M[\xi]$ and $\{\alpha_{R1,M_-,M_+,l}(\xi), \ell \in \{-M_-, \dots, M_+ - 1\}\} \in \mathbb{R}_{M-1}[\xi] \subset \mathbb{R}_M[\xi]$.

Proposition 3.10 (Identities on $\alpha_{R1,M_-,M_+,l}(\xi)$ (11a) and $\lambda_{R1,M_-,M_+,n}(\xi)$ (11a)). Assume the conditions and definitions of Proposition 2.2, and consider the stencil s_{i,M_-,M_+} (Definition 1.1) and its substencils (Definition 1.2) s_{i,M_-,M_+-1}

and S_{i,M_-,M_+} . The following identities hold $\forall \xi \in \mathbb{R}$

$$\alpha_{R_1,M_-,M_+,M_+}(\xi) = (-1)^{M_-} \alpha_{R_1,M_-,M_+,-M_-}(\xi) \quad (27a)$$

$$\alpha_{R_1,M_-,M_+-1,\ell}(\xi) \neq \alpha_{R_1,M_-,M_+,\ell}(\xi) \quad \forall \ell \in \{-M_- + 1, \dots, M_+ - 1\} \quad (27b)$$

$$\lambda_{R_1,M_-,M_+,M}(\xi) = (-1)^{M_-} \alpha_{R_1,M_-,M_+,-M_-}(\xi) \quad (27c)$$

$$\lambda_{R_1,M_-,M_+-1,M}(\xi) = -\alpha_{R_1,M_-,M_+,M}(\xi) \quad (27d)$$

$$0_{\mathbb{R}_{M-1}[\xi]}(\xi) \stackrel{(19b)}{\neq} \alpha_{R_1,M_-,M_+,M_+}(\xi) = \lambda_{R_1,M_-,M_+,M}(\xi) - \lambda_{R_1,M_-,M_+-1,M}(\xi) \quad (27e)$$

PROOF. Let $p(x) \in \mathbb{R}_M[x]$ be a polynomial of degree $\leq M$. Then by [7, Theorem 5.1, p. 296] its reconstruction pair $q(x) := [R_{(1;\Delta x)}(p)](x) \in \mathbb{R}_M[x]$, and $\deg(q) = \deg(p)$ (Remark 2.4).

Proof of (27a): By Proposition 2.2, $\forall p(x) \in \mathbb{R}_{M-1}[x]$, its reconstructing polynomials (Definition 1.4) on the 2 stencils $S_{i,M_-,M_+-1} := \{i - M_-, \dots, i + M_+ - 1\}$ and $S_{i,M_-,M_+} := \{i - M_- + 1, \dots, i + M_+\}$, which contain the same number of M points but are shifted by 1 cell, are exactly equal to the reconstruction pair of $p(x)$ (Definition 1.4) $q(x) := [R_{(1;\Delta x)}(p)](x)$, because of (8b), since $p(x) \in \mathbb{R}_{M-1}[x] \implies p^{(s)}(x) = 0_{\mathbb{R}_{M-1}[x]}(x) \forall s \geq M$. Hence, by (8a),

$$\begin{aligned} p_{R_1,M_-,M_+-1}(x_i + \xi \Delta x; x_i, \Delta x; p) &\stackrel{(8a)}{=} \sum_{\ell=-M_-}^{M_+-1} \alpha_{R_1,M_-,M_+-1,\ell}(\xi) p(x_i + \ell \Delta x) \stackrel{(8b)}{=} \\ &\stackrel{(8b)}{=} p_{R_1,M_-,M_+}(x_i + \xi \Delta x; x_i, \Delta x; p) \stackrel{(8a)}{=} \sum_{\ell=-M_+}^{M_+} \alpha_{R_1,M_-,M_+,\ell}(\xi) p(x_i + \ell \Delta x) \stackrel{(8b)}{=} \\ &\stackrel{(8b)}{=} q(x_i + \xi \Delta x) := [R_{(1;\Delta x)}(p)](x_i + \xi \Delta x) \end{aligned} \quad \begin{cases} \forall p(x) \in \mathbb{R}_{M-1}[x] \\ \forall \xi \in \mathbb{R} \\ \forall x_i \in \mathbb{R} \\ \forall \Delta x \in \mathbb{R}_{>0} \end{cases} \quad (28a)$$

whence

$$\begin{aligned} &\alpha_{R_1,M_-,M_+-1,-M_-}(\xi) p(x_i - M_- \Delta x) + \\ &\sum_{\ell=-M_+}^{M_+-1} (\alpha_{R_1,M_-,M_+-1,\ell}(\xi) - \alpha_{R_1,M_-,M_+,\ell}(\xi)) p(x_i + \ell \Delta x) - \\ &\alpha_{R_1,M_-,M_+,M_+}(\xi) p(x_i + M_+ \Delta x) = 0 \end{aligned} \quad \begin{cases} \forall p(x) \in \mathbb{R}_{M-1}[x] \\ \forall \xi \in \mathbb{R} \\ \forall x_i \in \mathbb{R} \\ \forall \Delta x \in \mathbb{R}_{>0} \end{cases} \quad (28b)$$

Applying (28b) to the polynomial

$$\mathbb{R}_{M-1}[x] \ni \prod_{m=-M_+}^{M_+-1} (x - x_i - m \Delta x) = 0 \quad \forall x \in \{x_i - (M_- - 1)\Delta x, \dots, x_i + (M_+ - 1)\Delta x\} \quad (28c)$$

yields

$$\begin{aligned} &\alpha_{R_1,M_-,M_+-1,-M_-}(\xi) \left(\prod_{m=-M_+}^{M_+-1} (-M_- - m) \right) + \\ &\sum_{\ell=-M_+}^{M_+-1} (\alpha_{R_1,M_-,M_+-1,\ell}(\xi) - \alpha_{R_1,M_-,M_+,\ell}(\xi)) \underbrace{\left(\prod_{m=-M_+}^{M_+-1} (\ell - m) \right)}_{=0 \forall \ell \in \{-M_+ + 1, \dots, M_+ - 1\}} - \\ &\alpha_{R_1,M_-,M_+,M_+}(\xi) \left(\prod_{m=-M_+}^{M_+-1} (+M_+ - m) \right) = 0 \end{aligned} \quad \begin{cases} \forall \xi \in \mathbb{R} \\ \forall \Delta x \in \mathbb{R}_{>0} \end{cases} \quad (28d)$$

ie

$$\alpha_{R_1, M_-, M_+ - 1, -M_-}(\xi) \left(\prod_{m=-M_-+1}^{M_+-1} (-M_- - m) \right) = \alpha_{R_1, M_- - 1, M_+, +M_+}(\xi) \left(\prod_{m=-M_-+1}^{M_+-1} (+M_+ - m) \right) \quad \forall \xi \in \mathbb{R} \quad (28e)$$

Since

$$\prod_{m=-M_-+1}^{M_+-1} (+M_+ - m) \stackrel{k:=M_+-m}{=} \prod_{k=M_-1}^1 k = (M-1)! \quad (28f)$$

$$\prod_{m=-M_-+1}^{M_+-1} (-M_- - m) \stackrel{k:=-M_- - m}{=} \prod_{k=-1}^{-(M-1)} k = (-1)^{M-1} (M-1)! \quad (28g)$$

using (28f, 28g) in (28d) proves (27a).

Proof of (27b): Applying (28b), successively for $k \in \{-M_- + 1, \dots, M_+ - 1\}$, to the polynomials

$$\mathbb{R}_{M-1}[x] \ni \prod_{\substack{m=-M_-+1 \\ m \neq k}}^{M_+} (x - x_i - m\Delta x) = 0 \quad \forall x \in \{x_i - (M_- - 1)\Delta x, \dots, x_i + M_+\Delta x\} \setminus \{x_i + k\Delta x\} \quad (29a)$$

yields

$$(\alpha_{R_1, M_-, M_+ - 1, k}(\xi) - \alpha_{R_1, M_- - 1, M_+, k}(\xi)) \prod_{\substack{m=-M_-+1 \\ m \neq k}}^{M_+} (k - m) = -\alpha_{R_1, M_-, M_+ - 1, -M_-}(\xi) \prod_{\substack{m=-M_-+1 \\ m \neq -M_-}}^{M_+} (-M_- - m) \quad \forall k \in \{-M_- + 1, \dots, M_+ - 1\} \quad (29b)$$

and using (28g)

$$\alpha_{R_1, M_-, M_+ - 1, k}(\xi) - \alpha_{R_1, M_- - 1, M_+, k}(\xi) = -\frac{(-1)^M M!}{\prod_{\substack{n=k+M_- - 1 \\ n \neq 0}}^{k-M_+} n} \alpha_{R_1, M_-, M_+ - 1, -M_-}(\xi) \stackrel{(19b)}{\neq} 0_{\mathbb{R}_{M-1}[\xi]}(\xi) \quad \forall k \in \{-M_- + 1, \dots, M_+ - 1\} \quad (29c)$$

proving (27b) by (19b).¹⁵

Proof of (27c, 27d): Notice first that by (11c), for $n = M + 1 \stackrel{(1b)}{=} M_- + M_+ + 1$,

$$\begin{aligned} \lambda_{R_1, M_-, M_+, M+1}(\xi) &\stackrel{(11c)}{=} \sum_{\ell=0}^{M+1-M-1} \mu_{R_1, M_-, M_+, M+1-\ell}(\xi) \frac{(-1)^{\ell+1}}{(\ell+1)!} \left(\left(\xi - \frac{1}{2} \right)^{\ell+1} - \left(\xi + \frac{1}{2} \right)^{\ell+1} \right) \\ &= \mu_{R_1, M_-, M_+, M+1}(\xi) \frac{-1}{1!} \left(\xi - \frac{1}{2} - \xi - \frac{1}{2} \right) \\ &= \mu_{R_1, M_-, M_+, M+1}(\xi) \quad \begin{cases} \forall \xi \in \mathbb{R} \\ \forall M_{\pm} \in \mathbb{Z} : M := M_- + M_+ \geq 1 \end{cases} \end{aligned} \quad (30a)$$

$$\prod_{\substack{m=-M_-+1 \\ m \neq k}}^{M_+} (k - m) \stackrel{n:=k-m}{=} \prod_{\substack{n=k+M_- - 1 \\ n \neq 0}}^{k-M_+} n \neq 0$$

where $\mu_{R_1, M_-, M_+, n}(\xi)$ is defined by (11b). By (30a)

$$\lambda_{R_1, M_-, M_+ - 1, M}(\xi) \stackrel{(30a)}{=} \mu_{R_1, M_-, M_+ - 1, M}(\xi) \quad (30b)$$

$$\lambda_{R_1, M_- - 1, M_+, M}(\xi) \stackrel{(30a)}{=} \mu_{R_1, M_- - 1, M_+, M}(\xi) \quad \begin{cases} \forall \xi \in \mathbb{R} \\ \forall M_{\pm} \in \mathbb{Z} : M := M_- + M_+ \geq 2 \end{cases} \quad (30c)$$

Since $\forall p(x) \in \mathbb{R}_M[x] \implies p^{(n)}(x) = 0_{\mathbb{R}_M[x]}(x) \quad \forall n \geq M + 1$, we have by (8a, 8b),

$$\sum_{\ell=-M_-}^{M_+-1} \alpha_{R_1, M_-, M_+ - 1, \ell}(\xi) p(x_i + \ell \Delta x) \stackrel{(8a, 8b)}{=} q(x_i + \xi \Delta x) + \mu_{R_1, M_-, M_+ - 1, M}(\xi) \Delta x^M p^{(M)}(x_i) \quad (30d)$$

$$\sum_{\ell=-M_-}^{M_+} \alpha_{R_1, M_-, M_+, \ell}(\xi) p(x_i + \ell \Delta x) \stackrel{(8a, 8b)}{=} q(x_i + \xi \Delta x) \quad \begin{cases} \forall p(x) \in \mathbb{R}_M[x] \\ q(x) = [R_{(1; \Delta x)}(p)](x) \\ \forall \xi \in \mathbb{R} \\ \forall x_i \in \mathbb{R} \\ \forall \Delta x \in \mathbb{R}_{>0} \end{cases} \quad (30e)$$

$$\sum_{\ell=-M_- - 1}^{M_+} \alpha_{R_1, M_- - 1, M_+, \ell}(\xi) p(x_i + \ell \Delta x) \stackrel{(8a, 8b)}{=} q(x_i + \xi \Delta x) + \mu_{R_1, M_- - 1, M_+, M}(\xi) \Delta x^M p^{(M)}(x_i) \quad (30f)$$

for the reconstructing polynomials $p_{R_1, M_-, M_+ - 1}(x_i + \xi \Delta x; x_i, \Delta x; p)$ (30d), $p_{R_1, M_-, M_+}(x_i + \xi \Delta x; x_i, \Delta x; p)$ (30e), and $p_{R_1, M_- - 1, M_+}(x_i + \xi \Delta x; x_i, \Delta x; p)$ (30f). Consider the polynomials

$$\mathbb{R}_M[x] \ni \prod_{m=-M_-}^{M_+-1} (x - x_i - m \Delta x) = 0 \quad \forall x \in \{x_i - M_- \Delta x, \dots, x_i + (M_+ - 1) \Delta x\} \quad (30g)$$

$$\mathbb{R}_M[x] \ni \prod_{m=-M_- + 1}^{M_+} (x - x_i - m \Delta x) = 0 \quad \forall x \in \{x_i - (M_- - 1) \Delta x, \dots, x_i + M_+ \Delta x\} \quad (30h)$$

Obviously,

$$\frac{d^M}{dx^M} \left(\prod_{m=-M_-}^{M_+-1} (x - x_i - m \Delta x) \right) = \frac{d^M}{dx^M} \left(\prod_{m=-M_- + 1}^{M_+} (x - x_i - m \Delta x) \right) = M! \quad \begin{cases} \forall x \in \mathbb{R} \\ \forall x_i \in \mathbb{R} \\ \forall \Delta x \in \mathbb{R}_{>0} \end{cases} \quad (30i)$$

Applying (30e) to the polynomials (30g, 30h), (30d) to the polynomial (30g), and (30f) to the polynomial (30h), we have, using (30i),

$$(30e, 30g) \implies \alpha_{R_1, M_-, M_+ + M_+}(\xi) \Delta x^M \prod_{m=-M_-}^{M_+-1} (M_+ - m) = R_{(1; \Delta x)} \left(\prod_{m=-M_-}^{M_+-1} (x - x_i - m \Delta x) \right) \quad (30j)$$

$$(30e, 30h) \implies \alpha_{R_1, M_-, M_+, -M_-}(\xi) \Delta x^M \prod_{m=-M_- + 1}^{M_+} (-M_- - m) = R_{(1; \Delta x)} \left(\prod_{m=-M_- + 1}^{M_+} (x - x_i - m \Delta x) \right) \quad (30k)$$

$$(30d, 30g) \implies 0 = R_{(1; \Delta x)} \left(\prod_{m=-M_-}^{M_+-1} (x - x_i - m \Delta x) \right) + \mu_{R_1, M_-, M_+ - 1, M}(\xi) \Delta x^M M! \quad (30l)$$

$$(30f, 30h) \implies 0 = R_{(1; \Delta x)} \left(\prod_{m=-M_- + 1}^{M_+} (x - x_i - m \Delta x) \right) + \mu_{R_1, M_- - 1, M_+, M}(\xi) \Delta x^M M! \quad (30m)$$

$$\forall x \in \mathbb{R} \quad \xi \Delta x := x - x_i \quad \forall x_i \in \mathbb{R} \quad \forall \Delta x \in \mathbb{R}_{>0}$$

and combining (30j) with (30l), and (30k) with (30m), we have

$$\alpha_{R_1, M_-, M_+, +M_+}(\xi) \prod_{m=-M_-}^{M_+-1} (M_+ - m) \stackrel{(30j, 30l)}{=} -\mu_{R_1, M_-, M_+-1, M}(\xi) M! \quad \forall \xi \in \mathbb{R} \quad (30n)$$

$$\alpha_{R_1, M_-, M_+, -M_-}(\xi) \prod_{m=-M_-+1}^{M_+} (-M_- - m) \stackrel{(30k, 30m)}{=} -\mu_{R_1, M_-, -1, M_+, M}(\xi) M! \quad \forall \xi \in \mathbb{R} \quad (30o)$$

which¹⁶ by (30b, 30c) prove (27c, 27d).

Proof of (27e): Applying (30f) to the polynomial (30g) yields

$$\alpha_{R_1, M_-, M_+, +M_+}(\xi) \Delta x^M \prod_{m=-M_-}^{M_+-1} (M_+ - m) \stackrel{(30f, 30g)}{=} R_{(1; \Delta x)} \left(\prod_{m=-M_-}^{M_+-1} (x - x_i - m \Delta x) \right) + \mu_{R_1, M_-, -1, M_+, M}(\xi) \Delta x^M M! \quad (31a)$$

$\forall x \in \mathbb{R} \quad \xi \Delta x := x - x_i \quad \forall x_i \in \mathbb{R} \quad \forall \Delta x \in \mathbb{R}_{>0}$

Combining (31a) and (30l) yields¹⁶

$$\alpha_{R_1, M_-, M_+, +M_+}(\xi) \stackrel{(31a, 30l)}{=} -\mu_{R_1, M_-, M_+-1, M}(\xi) + \mu_{R_1, M_-, -1, M_+, M}(\xi) \quad \forall \xi \in \mathbb{R} \quad (31b)$$

which by (30b, 30c) proves (27e). \square

4. Reconstruction by combination of substencils

4.1. Substencils of S_{i, M_-, M_+}

WENO reconstruction [3] on S_{i, M_-, M_+} achieves high-order in smooth regions and monotonicity near discontinuities by a nonlinear (depending on the values $f_{i+\ell}$ of $f(x)$ on the points of the stencil S_{i, M_-, M_+}) combination of reconstructions on substencils whose union equals the stencil.¹⁷ Central to this development is the determination of the underlying optimal (linear in f in the sense that the weight-functions depend only on x and not on f) combination of the reconstructing polynomials on the substencils to exactly obtain the reconstructing polynomial of the entire stencil.

Example 4.1 (Substencils (Definition 1.2)). Notice that a given stencil S_{i, M_-, M_+} can be divided into different families of substencils (Definition 1.2), depending on the chosen value of $K_s \leq M - 1$ (2b). The 0-level of subdivision ($K_s = 0$) corresponds to the original stencil, without subdivision. The $(M - 1)$ -level of subdivision ($K_s = M - 1$) corresponds to the subdivision of the original stencil to $K_s + 1 = M$ substencils of length equal to 1 cell, *ie* to the substencils $\{S_{i, M_-, M_+-1}, \dots, S_{i, M_+-1, M_+}\}$, on each of which polynomial interpolation, and as a consequence polynomial reconstruction (Remark 2.4), are of degree 1 (linear). As an example, we consider the successive subdivisions of the stencil $S_{i, 3, 3}$ (Fig. 3) which corresponds to a stencil symmetric around point i , and of the stencil $S_{i, 3, 4}$ (Fig. 4) which corresponds to a stencil symmetric around point $i + \frac{1}{2}$. We called the substencils of Definition 1.2 Neville substencils [12] because they are those used in the Neville algorithm [8, pp. 207–208] for the recursive construction of the interpolating polynomial. \square

¹⁶By analogy with (28f, 28g) we have

$$\prod_{m=-M_-}^{M_+-1} (+M_+ - m) \stackrel{k:=M_+-m}{=} \prod_{k=M}^1 k = M! \quad ; \quad \prod_{m=-M_-+1}^{M_+} (-M_- - m) \stackrel{k:=-M_- - m}{=} \prod_{k=-1}^{-M} k = (-1)^M M!$$

¹⁷ Shu [5, 3] uses the terms big stencil and small stencils to denote the stencil and its substencils.

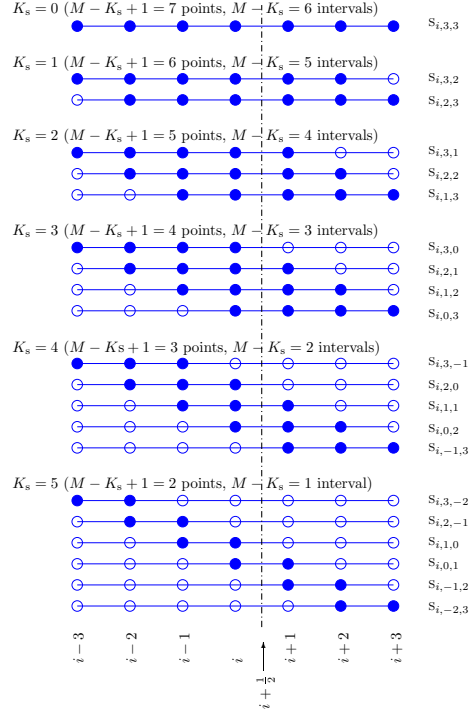


Figure 3: Successive subdivisions of the stencil $S_{i,3,3}$, for different values of $K_s \in \{0, \dots, M-1=5\}$ (Definition 1.2).

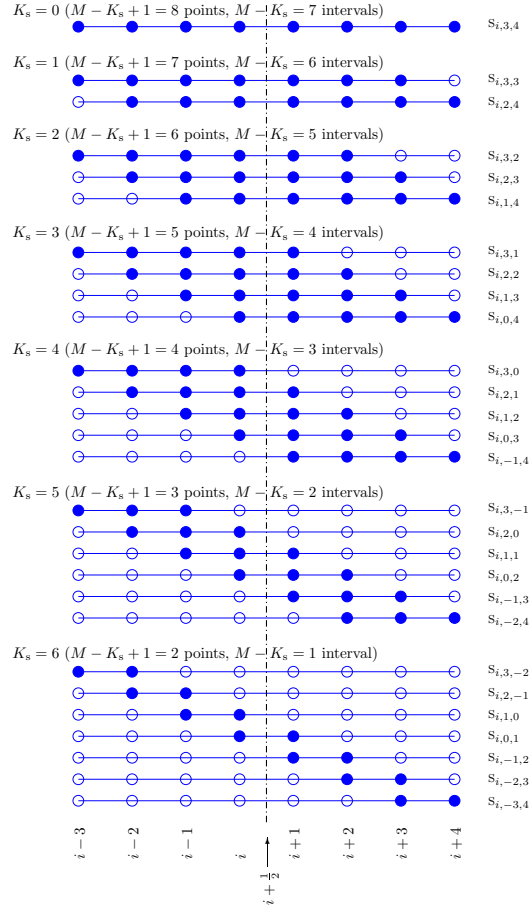


Figure 4: Successive subdivisions of the stencil $S_{i,3,4}$, for different values of $K_s \in \{0, \dots, M-1=6\}$ (Definition 1.2).

4.2. ($K_s = 1$)-level subdivision

The starting point for developing a recursive formulation for the weight-functions is to consider the ($K_s = 1$)-level subdivision of S_{i,M_-,M_+} . The resulting substencils S_{i,M_-,M_+-1} and S_{i,M_-1,M_+} have equal lengths of $M - 1$ cells (M points), but are shifted by 1 cell (Figs. 3, 4). If a ($K_s = 1$)-level subdivision rule can be established, then it can be readily extended to ($K_s > 1$)-levels using the general recurrence relation proven in [12, Lemma 2.1].

Lemma 4.2 (Rational weight-functions for ($K_s = 1$)-level subdivision). *Assume the conditions and definitions of Proposition 2.2, and consider the stencil S_{i,M_-,M_+} and its substencils (Definition 1.2) S_{i,M_-,M_+-1} and S_{i,M_-1,M_+} . Define the functions $\sigma_{R_1,M_-,M_+,1,0}(\xi)$ and $\sigma_{R_1,M_-,M_+,1,1}(\xi)$ by*

$$\sigma_{R_1,M_-,M_+,1,0}(\xi) := \frac{\alpha_{R_1,M_-,M_+,-M_-}(\xi)}{\alpha_{R_1,M_-,M_+-1,-M_-}(\xi)} \stackrel{(27)}{=} \frac{\lambda_{R_1,M_-1,M_+,M}(\xi)}{\lambda_{R_1,M_-1,M_+,M}(\xi) - \lambda_{R_1,M_-,M_+-1,M}(\xi)} \quad (32a)$$

$$\sigma_{R_1,M_-,M_+,1,1}(\xi) := \frac{\alpha_{R_1,M_-,M_+,M_+}(\xi)}{\alpha_{R_1,M_-1,M_+,M_+}(\xi)} \stackrel{(27)}{=} \frac{\lambda_{R_1,M_-,M_+-1,M}(\xi)}{\lambda_{R_1,M_-,M_+-1,M}(\xi) - \lambda_{R_1,M_-1,M_+,M}(\xi)} \quad (32b)$$

satisfying the consistency condition

$$\sigma_{R_1,M_-,M_+,1,0}(\xi) + \sigma_{R_1,M_-,M_+,1,1}(\xi) = 1 \quad (32c)$$

Then the reconstructing polynomial on S_{i,M_-,M_+} (Proposition 2.2) can be constructed by combination of the reconstructing polynomials on the 2 ($K_s = 1$)-level-subdivision substencils as

$$\begin{aligned} & p_{R_1,M_-,M_+}(x_i + \xi\Delta x; x_i, \Delta x; f) = \\ &= \frac{\lambda_{R_1,M_-1,M_+,M}(\xi) p_{R_1,M_-,M_+-1}(x_i + \xi\Delta x; x_i, \Delta x; f) - \lambda_{R_1,M_-,M_+-1,M}(\xi) p_{R_1,M_-1,M_+}(x_i + \xi\Delta x; x_i, \Delta x; f)}{\lambda_{R_1,M_-1,M_+,M}(\xi) - \lambda_{R_1,M_-,M_+-1,M}(\xi)} \quad (32d) \\ & \forall \xi \in \mathbb{R} \quad \forall x_i \in \mathbb{R} \quad \forall \Delta x \in \mathbb{R}_{>0} \quad \forall f : \mathbb{R} \rightarrow \mathbb{R} \end{aligned}$$

and can be represented, almost everywhere, as

$$\begin{aligned} p_{R_1,M_-,M_+}(x_i + \xi\Delta x; x_i, \Delta x; f) &= \sigma_{R_1,M_-,M_+,1,0}(\xi) p_{R_1,M_-,M_+-1}(x_i + \xi\Delta x; x_i, \Delta x; f) \\ &+ \sigma_{R_1,M_-,M_+,1,1}(\xi) p_{R_1,M_-1,M_+}(x_i + \xi\Delta x; x_i, \Delta x; f) \quad (32e) \end{aligned}$$

$$\forall \xi \in \mathbb{R} \setminus \{\xi \in \mathbb{R} : \alpha_{R_1,M_-1,M_+,M_+}(\xi) = 0\} \quad \forall x_i \in \mathbb{R} \quad \forall \Delta x \in \mathbb{R}_{>0} \quad \forall f : \mathbb{R} \rightarrow \mathbb{R}$$

The functions $\sigma_{R_1,M_-,M_+,1,0}(\xi)$ and $\sigma_{R_1,M_-,M_+,1,1}(\xi)$ satisfying (32c, 32e) are unique.

PROOF. By Proposition 3.10 we have

$$\frac{\alpha_{R_1,M_-,M_+,-M_-}(\xi)}{\alpha_{R_1,M_-,M_+-1,-M_-}(\xi)} \stackrel{(27a, 27c)}{=} \frac{(-1)^{M-1} \lambda_{R_1,M_-1,M_+,M}(\xi)}{(-1)^{M-1} \alpha_{R_1,M_-1,M_+,M_+}(\xi)} \stackrel{(27e)}{=} \frac{\lambda_{R_1,M_-1,M_+,M}(\xi)}{\lambda_{R_1,M_-1,M_+,M}(\xi) - \lambda_{R_1,M_-,M_+-1,M}(\xi)} \quad (33a)$$

proving (32a), and

$$\frac{\alpha_{R_1,M_-,M_+,M_+}(\xi)}{\alpha_{R_1,M_-1,M_+,M_+}(\xi)} \stackrel{(27d, 27e)}{=} \frac{-\lambda_{R_1,M_-,M_+-1,M}(\xi)}{\lambda_{R_1,M_-1,M_+,M}(\xi) - \lambda_{R_1,M_-,M_+-1,M}(\xi)} \quad (33b)$$

proving (32b). Obviously (32c) holds because

$$\begin{aligned} \sigma_{R_1,M_-,M_+,1,0}(\xi) + \sigma_{R_1,M_-,M_+,1,1}(\xi) &\stackrel{(32a, 32b)}{=} \frac{\lambda_{R_1,M_-1,M_+,M}(\xi)}{\lambda_{R_1,M_-1,M_+,M}(\xi) - \lambda_{R_1,M_-,M_+-1,M}(\xi)} \\ &+ \frac{-\lambda_{R_1,M_-,M_+-1,M}(\xi)}{\lambda_{R_1,M_-1,M_+,M}(\xi) - \lambda_{R_1,M_-,M_+-1,M}(\xi)} \\ &= \frac{\lambda_{R_1,M_-1,M_+,M}(\xi) - \lambda_{R_1,M_-,M_+-1,M}(\xi)}{\lambda_{R_1,M_-1,M_+,M}(\xi) - \lambda_{R_1,M_-,M_+-1,M}(\xi)} = 1 \quad (33c) \end{aligned}$$

Proof of (32e): Let $p(x) \in \mathbb{R}_M[x]$ (16). Then, by (16), $q(x) := [R_{(1;\Delta x)}(p)](x) \in \mathbb{R}_M[x]$. Since $\forall q(x) \in \mathbb{R}_M[x] \implies q^{(n)}(x) = 0_{\mathbb{R}_M[x]} \forall n \geq M+1$, we have, by application of (8c), and taking into account Remark 2.4,

$$p_{R_1, M_-, M_+ - 1}(x_i + \xi \Delta x; x_i, \Delta x; p) \stackrel{(8c, 16)}{=} q(x_i + \xi \Delta x) + \lambda_{R_1, M_-, M_+ - 1, M}(\xi) \Delta x^M q^{(M)}(x_i + \xi \Delta x) \quad (34a)$$

$$p_{R_1, M_-, M_+}(x_i + \xi \Delta x; x_i, \Delta x; p) \stackrel{(8c, 16)}{=} q(x_i + \xi \Delta x) \quad \left\{ \begin{array}{l} \forall p(x) \in \mathbb{R}_M[x] \\ q(x) := [R_{(1;\Delta x)}(p)](x) \quad (16) \\ \forall \xi \in \mathbb{R} \\ \forall x_i \in \mathbb{R} \\ \forall \Delta x \in \mathbb{R}_{>0} \end{array} \right. \quad (34b)$$

$$p_{R_1, M_-, 1, M_+}(x_i + \xi \Delta x; x_i, \Delta x; p) \stackrel{(8c, 16)}{=} q(x_i + \xi \Delta x) + \lambda_{R_1, M_-, 1, M_+, M}(\xi) \Delta x^M q^{(M)}(x_i + \xi \Delta x) \quad (34c)$$

Combining (34a) weighted by (32a), and (34c) weighted by (32b) yields

$$\begin{aligned} & \sigma_{R_1, M_-, M_+, 1, 0}(\xi) p_{R_1, M_-, M_+ - 1}(x_i + \xi \Delta x; x_i, \Delta x; p) + \sigma_{R_1, M_-, M_+, 1, 1}(\xi) p_{R_1, M_-, 1, M_+}(x_i + \xi \Delta x; x_i, \Delta x; p) = \\ & \stackrel{(34a, 34c)}{=} (\sigma_{R_1, M_-, M_+, 1, 0}(\xi) + \sigma_{R_1, M_-, M_+, 1, 1}(\xi)) q(x_i + \xi \Delta x) \\ & + (\sigma_{R_1, M_-, M_+, 1, 0}(\xi) \lambda_{R_1, M_-, M_+ - 1, M}(\xi) + \sigma_{R_1, M_-, M_+, 1, 1}(\xi) \lambda_{R_1, M_-, 1, M_+, M}(\xi)) \Delta x^M q^{(M)}(x_i + \xi \Delta x) \\ & \stackrel{(32a-32c)}{=} q(x_i + \xi \Delta x) \\ & + \left(\frac{\lambda_{R_1, M_-, 1, M_+, M}(\xi) \lambda_{R_1, M_-, M_+ - 1, M}(\xi)}{\lambda_{R_1, M_-, 1, M_+, M}(\xi) - \lambda_{R_1, M_-, M_+ - 1, M}(\xi)} + \frac{-\lambda_{R_1, M_-, M_+ - 1, M}(\xi) \lambda_{R_1, M_-, 1, M_+, M}(\xi)}{\lambda_{R_1, M_-, 1, M_+, M}(\xi) - \lambda_{R_1, M_-, M_+ - 1, M}(\xi)} \right) \Delta x^M q^{(M)}(x_i + \xi \Delta x) \\ & = q(x_i + \xi \Delta x) \stackrel{(34b)}{=} p_{R_1, M_-, M_+}(x_i + \xi \Delta x; x_i, \Delta x; p) \quad \left\{ \begin{array}{l} \forall p(x) \in \mathbb{R}_M[x] \\ q(x) := [R_{(1;\Delta x)}(p)](x) \quad (16) \\ \forall \xi \in \mathbb{R} \\ \forall x_i \in \mathbb{R} \\ \forall \Delta x \in \mathbb{R}_{>0} \end{array} \right. \quad (34d) \end{aligned}$$

$$\begin{aligned} & \implies \sigma_{R_1, M_-, M_+, 1, 0}(\xi) p_{R_1, M_-, M_+ - 1}(x_i + \xi \Delta x; x_i, \Delta x; p) + \\ & \sigma_{R_1, M_-, M_+, 1, 1}(\xi) p_{R_1, M_-, 1, M_+}(x_i + \xi \Delta x; x_i, \Delta x; p) = p_{R_1, M_-, M_+}(x_i + \xi \Delta x; x_i, \Delta x; p) \quad (34e) \\ & \forall p(x) \in \mathbb{R}_M[x] \quad \forall \xi \in \mathbb{R} \setminus \{\xi \in \mathbb{R} : \alpha_{R_1, M_-, 1, M_+, M}(\xi) = 0\} \quad \forall x_i \in \mathbb{R} \quad \forall \Delta x \in \mathbb{R}_{>0} \end{aligned}$$

which shows that (32e) is valid $\forall f(x) \in \mathbb{R}_M[x]$. Using the representation (8a) of the reconstructing polynomial in (34e)

$$\begin{aligned} 0 & \stackrel{(34e, 8a)}{=} (\alpha_{R_1, M_-, M_+, -M_-}(\xi) - \sigma_{R_1, M_-, M_+, 1, 0}(\xi) \alpha_{R_1, M_-, M_+ - 1, -M_-}(\xi)) p(x_i - M_- \Delta x) \\ & + \sum_{\ell=-M_-+1}^{M_+-1} (\alpha_{R_1, M_-, M_+, \ell}(\xi) - \sigma_{R_1, M_-, M_+, 1, 0}(\xi) \alpha_{R_1, M_-, M_+ - 1, \ell}(\xi) - \sigma_{R_1, M_-, M_+, 1, 1}(\xi) \alpha_{R_1, M_-, 1, M_+, \ell}(\xi)) p(x_i + \ell \Delta x) \\ & + (\alpha_{R_1, M_-, M_+, +M_+}(\xi) - \sigma_{R_1, M_-, M_+, 1, 1}(\xi) \alpha_{R_1, M_-, 1, M_+, +M_+}(\xi)) p(x_i + M_+ \Delta x) \\ & \stackrel{(32a, 32b)}{=} \sum_{\ell=-M_-+1}^{M_+-1} (\alpha_{R_1, M_-, M_+, \ell}(\xi) - \sigma_{R_1, M_-, M_+, 1, 0}(\xi) \alpha_{R_1, M_-, M_+ - 1, \ell}(\xi) - \sigma_{R_1, M_-, M_+, 1, 1}(\xi) \alpha_{R_1, M_-, 1, M_+, \ell}(\xi)) p(x_i + \ell \Delta x) \\ & \forall p(x) \in \mathbb{R}_M[x] \quad \forall \xi \in \mathbb{R} \setminus \{\xi \in \mathbb{R} : \alpha_{R_1, M_-, 1, M_+, M}(\xi) = 0\} \quad \forall x_i \in \mathbb{R} \quad \forall \Delta x \in \mathbb{R}_{>0} \quad (34f) \end{aligned}$$

where we used

$$\alpha_{R_1, M_-, M_+, -M_-}(\xi) \stackrel{(32a)}{=} \sigma_{R_1, M_-, M_+, 1, 0}(\xi) \alpha_{R_1, M_-, M_+ - 1, -M_-}(\xi) \quad \forall \xi \in \mathbb{R} \quad (34g)$$

$$\alpha_{R_1, M_-, M_+, +M_+}(\xi) \stackrel{(32b)}{=} \sigma_{R_1, M_-, M_+, 1, 1}(\xi) \alpha_{R_1, M_-, 1, M_+, +M_+}(\xi) \quad \forall \xi \in \mathbb{R} \quad (34h)$$

Applying (34f) successively to the polynomials

$$\mathbb{R}_M[x] \ni \prod_{\substack{m=-M_- \\ m \neq k}}^{M_+} (x - x_i - m\Delta x) = 0 \quad \forall x \in \{x_i - M_- \Delta x, \dots, x_i + M_+ \Delta x\} \setminus \{x_i + k\Delta x\} \quad (34i)$$

yields

$$\begin{aligned} \alpha_{R_1, M_-, M_+, \ell}(\xi) &= \sigma_{R_1, M_-, M_+, 1, 0}(\xi) \alpha_{R_1, M_-, M_+ - 1, \ell}(\xi) \\ &+ \sigma_{R_1, M_-, M_+, 1, 1}(\xi) \alpha_{R_1, M_-, M_+, \ell}(\xi) \quad \begin{cases} \forall \xi \in \mathbb{R} \setminus \{\xi \in \mathbb{R} : \alpha_{R_1, M_-, M_+, M_+}(\xi) = 0\} \\ \forall \ell \in \{-M_- + 1, \dots, M_+ - 1\} \end{cases} \end{aligned} \quad (34j)$$

Combining the representation (8a) of the reconstructing polynomial with (34g, 34h, 34j) proves (32e), $\forall f : \mathbb{R} \rightarrow \mathbb{R}$.

Proof of (32d): Obviously, the functions $\sigma_{R_1, M_-, M_+, 1, 0}(\xi)$ and $\sigma_{R_1, M_-, M_+, 1, 1}(\xi)$ are defined everywhere ($\forall \xi \in \mathbb{R}$) except at

$$\{\xi_{R_1, M_-, M_+, n}; n \in \{-M_- + 1, \dots, M_+ - 1\}\} \quad (35a)$$

$$\stackrel{(24)}{=} \{\xi \in \mathbb{R} : \alpha_{R_1, M_-, M_+, M_+}(\xi) = 0\} \quad (35b)$$

$$\stackrel{(27a)}{=} \{\xi \in \mathbb{R} : \alpha_{R_1, M_-, M_+ - 1, -M_-}(\xi) = 0\} \quad (35c)$$

$$\stackrel{(27e)}{=} \{\xi \in \mathbb{R} : \lambda_{R_1, M_-, M_+, M}(\xi) - \lambda_{R_1, M_-, M_+ - 1, M}(\xi) = 0\} \quad (35d)$$

Recall that (Proposition 3.5) all of the $M - 1$ roots of the polynomial $\alpha_{R_1, M_-, M_+, M}(\xi)$ are real (24). However, using (27a, 27b) in (32e) yields

$$\begin{aligned} & p_{R_1, M_-, M_+}(x_i + \xi \Delta x; x_i, \Delta x; f) \stackrel{(32e, 27a, 27b)}{=} \\ &= \frac{\lambda_{R_1, M_-, M_+, M}(\xi) p_{R_1, M_-, M_+ - 1}(x_i + \xi \Delta x; x_i, \Delta x; f) - \lambda_{R_1, M_-, M_+ - 1, M}(\xi) p_{R_1, M_-, M_+}(x_i + \xi \Delta x; x_i, \Delta x; f)}{\lambda_{R_1, M_-, M_+, M}(\xi) - \lambda_{R_1, M_-, M_+ - 1, M}(\xi)} \quad (35e) \\ & \forall \xi \in \mathbb{R} \setminus \{\xi \in \mathbb{R} : \alpha_{R_1, M_-, M_+, M}(\xi) = 0\} \quad \forall x_i \in \mathbb{R} \quad \forall \Delta x \in \mathbb{R}_{>0} \quad \forall f : \mathbb{R} \rightarrow \mathbb{R} \end{aligned}$$

or equivalently, using the representation (8a) of the reconstructing polynomial in (35e)

$$\begin{aligned} & p_{R_1, M_-, M_+}(x_i + \xi \Delta x; x_i, \Delta x; f) \stackrel{(35e, 8a)}{=} \\ & \frac{\lambda_{R_1, M_-, M_+, M}(\xi) \alpha_{R_1, M_-, M_+ - 1, -M_-}(\xi)}{\lambda_{R_1, M_-, M_+, M}(\xi) - \lambda_{R_1, M_-, M_+ - 1, M}(\xi)} f(x_i - M_- \Delta x) \\ & + \sum_{\ell=-M_-+1}^{M_+-1} \frac{\lambda_{R_1, M_-, M_+, M}(\xi) \alpha_{R_1, M_-, M_+ - 1, \ell}(\xi) - \lambda_{R_1, M_-, M_+ - 1, M}(\xi) \alpha_{R_1, M_-, M_+, \ell}(\xi)}{\lambda_{R_1, M_-, M_+, M}(\xi) - \lambda_{R_1, M_-, M_+ - 1, M}(\xi)} f(x_i + \ell \Delta x) \\ & + \frac{-\lambda_{R_1, M_-, M_+ - 1, M}(\xi) \alpha_{R_1, M_-, M_+, M_+}(\xi)}{\lambda_{R_1, M_-, M_+, M}(\xi) - \lambda_{R_1, M_-, M_+ - 1, M}(\xi)} f(x_i + M_+ \Delta x) \quad (35f) \\ & \forall \xi \in \mathbb{R} \setminus \{\xi \in \mathbb{R} : \alpha_{R_1, M_-, M_+, M}(\xi) = 0\} \quad \forall x_i \in \mathbb{R} \quad \forall \Delta x \in \mathbb{R}_{>0} \quad \forall f : \mathbb{R} \rightarrow \mathbb{R} \end{aligned}$$

To prove (32d) by (35f) we need to show that it is valid $\forall \xi \in \mathbb{R}$. Rewriting (33a, 33b) we have

$$\begin{aligned} \alpha_{R_1, M_-, M_+, -M_-}(\xi) & \stackrel{(33a)}{=} \frac{\lambda_{R_1, M_-, M_+, M}(\xi) \alpha_{R_1, M_-, M_+ - 1, -M_-}(\xi)}{\lambda_{R_1, M_-, M_+, M}(\xi) - \lambda_{R_1, M_-, M_+ - 1, M}(\xi)} \quad \forall \xi \in \mathbb{R} \implies \\ & (\lambda_{R_1, M_-, M_+, M}(\xi) - \lambda_{R_1, M_-, M_+ - 1, M}(\xi)) \mid (\lambda_{R_1, M_-, M_+, M}(\xi) \alpha_{R_1, M_-, M_+ - 1, -M_-}(\xi)) \quad (35g) \end{aligned}$$

$$\begin{aligned} \alpha_{R_1, M_-, M_+, M_+}(\xi) & \stackrel{(33a)}{=} \frac{-\lambda_{R_1, M_-, M_+ - 1, M}(\xi) \alpha_{R_1, M_-, M_+, M_+}(\xi)}{\lambda_{R_1, M_-, M_+, M}(\xi) - \lambda_{R_1, M_-, M_+ - 1, M}(\xi)} \quad \forall \xi \in \mathbb{R} \implies \\ & (\lambda_{R_1, M_-, M_+, M}(\xi) - \lambda_{R_1, M_-, M_+ - 1, M}(\xi)) \mid (\lambda_{R_1, M_-, M_+ - 1, M}(\xi) \alpha_{R_1, M_-, M_+, M_+}(\xi)) \quad (35h) \end{aligned}$$

Recall that by (11c) $\deg(\lambda_{R_1, M_-, M_+ - 1, M}) = \deg(\lambda_{R_1, M_-, M_+, M}) = M$ [7, Proposition 4.7, p. 294], by (11a) $\deg(\alpha_{R_1, M_-, M_+ - 1, -M_-}) = \deg(\alpha_{R_1, M_-, M_+, M_+}) = M - 1$ and $\deg(\alpha_{R_1, M_-, M_+, -M_-}) = \deg(\alpha_{R_1, M_-, M_+, M_+}) = M$, and by (27e) $\deg(\lambda_{R_1, M_-, M_+, M} - \lambda_{R_1, M_-, M_+ - 1, M}) = \deg(\alpha_{R_1, M_-, M_+, M_+}) = M - 1$.

Using again (27a, 27b) in (34j) yields

$$\begin{aligned} & \forall \xi \in \mathbb{R} \setminus \{\xi \in \mathbb{R} : \alpha_{R_1, M_-, M_+, M_+}(\xi) = 0\} \quad \forall \ell \in \{-M_- + 1, \dots, M_+ - 1\} \\ & \alpha_{R_1, M_-, M_+, \ell}(\xi) \stackrel{(27a, 27b, 34j)}{=} \frac{\lambda_{R_1, M_-, M_+, M}(\xi) \alpha_{R_1, M_-, M_+ - 1, \ell}(\xi) - \lambda_{R_1, M_-, M_+ - 1, M}(\xi) \alpha_{R_1, M_-, M_+, \ell}(\xi)}{\lambda_{R_1, M_-, M_+, M}(\xi) - \lambda_{R_1, M_-, M_+ - 1, M}(\xi)} \end{aligned} \quad (35i)$$

Since the set $\{\xi \in \mathbb{R} : \alpha_{R_1, M_-, M_+, M_+}(\xi) = 0\}$ contains only $M - 1$ isolated points (Proposition 3.5), the result of the polynomial division (35i) must be valid $\forall \xi \in \mathbb{R}$, implying

$$\begin{aligned} & (\lambda_{R_1, M_-, M_+, M}(\xi) - \lambda_{R_1, M_-, M_+ - 1, M}(\xi)) \mid \\ & (\lambda_{R_1, M_-, M_+, M}(\xi) \alpha_{R_1, M_-, M_+ - 1, \ell}(\xi) - \lambda_{R_1, M_-, M_+ - 1, M}(\xi) \alpha_{R_1, M_-, M_+, \ell}(\xi)) \\ & \forall \ell \in \{-M_- + 1, \dots, M_+ - 1\} \end{aligned} \quad (35j)$$

By (35g, 35h, 35j), we have that (35f) is valid $\forall \xi \in \mathbb{R}$, proving (32d).

Proof of uniqueness: We have proved existence by construction, $\forall \xi \in \mathbb{R} \setminus \{\xi \in \mathbb{R} : \alpha_{R_1, M_-, M_+, M_+}(\xi) = 0\}$, of rational weighting functions, $\sigma_{R_1, M_-, M_+, 1, 0}(\xi)$ (32a) and $\sigma_{R_1, M_-, M_+, 1, 1}(\xi)$ (32b), satisfying the consistency relation (32c), which combine the reconstructing polynomials on the substencils (Definition 1.2) $S_{i, M_-, M_+ - 1}$ and S_{i, M_-, M_+} into the reconstructing polynomial on S_{i, M_-, M_+} (32e). To prove uniqueness, recall that by (32c) $\sigma_{R_1, M_-, M_+, 1, 1}(\xi) = 1 - \sigma_{R_1, M_-, M_+, 1, 0}(\xi)$, and rewrite (32e) as

$$\begin{aligned} p_{R_1, M_-, M_+}(x_i + \xi \Delta x; x_i, \Delta x; f) & \stackrel{(32c, 32e)}{=} \sigma_{R_1, M_-, M_+, 1, 0}(\xi) (p_{R_1, M_-, M_+ - 1}(x_i + \xi \Delta x; x_i, \Delta x; f) \\ & \quad - p_{R_1, M_-, M_+}(x_i + \xi \Delta x; x_i, \Delta x; f)) \\ & \quad + p_{R_1, M_-, M_+}(x_i + \xi \Delta x; x_i, \Delta x; f) \end{aligned} \quad (36a)$$

$$\forall \xi \in \mathbb{R} \setminus \{\xi \in \mathbb{R} : \alpha_{R_1, M_-, M_+, M_+}(\xi) = 0\} \quad \forall x_i \in \mathbb{R} \quad \forall \Delta x \in \mathbb{R}_{>0} \quad \forall f : \mathbb{R} \longrightarrow \mathbb{R}$$

Hence, assuming the existence of 2 different weight-functions $[\sigma_{R_1, M_-, M_+, 1, 0}]_A(\xi) \neq [\sigma_{R_1, M_-, M_+, 1, 0}]_B(\xi)$ satisfying (36a) $\forall \xi \in \mathbb{R} \forall f : \mathbb{R} \longrightarrow \mathbb{R}$ would imply $p_{R_1, M_-, M_+ - 1}(x_i + \xi \Delta x; x_i, \Delta x; f) = p_{R_1, M_-, M_+}(x_i + \xi \Delta x; x_i, \Delta x; f) \forall \xi \in \mathbb{R} \forall f : \mathbb{R} \longrightarrow \mathbb{R}$. This is obviously a contradiction, since, by Proposition 2.2, the 2 polynomials $p_{R_1, M_-, M_+ - 1}(x_i + \xi \Delta x; x_i, \Delta x; f)$ and $p_{R_1, M_-, M_+}(x_i + \xi \Delta x; x_i, \Delta x; f)$ are defined by 2 different sets of values, $\{f_{i-M_-}, \dots, f_{i+M_+ - 1}\}$ and $\{f_{i-M_- + 1}, \dots, f_{i+M_+}\}$, respectively.¹⁸ \square

Remark 4.3 ((32d) vs (32e)). The expression (32d) of the Lagrange reconstructing polynomial $p_{R_1, M_-, M_+}(x_i + \xi \Delta x; x_i, \Delta x; f)$ on S_{i, M_-, M_+} is valid $\forall \xi \in \mathbb{R}$, because the rational expression (32d) yields exactly $p_{R_1, M_-, M_+}(x_i + \xi \Delta x; x_i, \Delta x; f)$ by polynomial division. On the other hand, the weight functions $\sigma_{R_1, M_-, M_+, 1, 0}(\xi)$ (32a) and $\sigma_{R_1, M_-, M_+, 1, 1}(\xi)$ (32b), are not defined at the poles of the rational expressions (32a, 32b), where the representation (32e) is not possible. \square

Corollary 4.4 (Identities for $(K_s = 1)$ -level subdivision). *Assume the conditions and definitions of Lemma 4.2. Then the following identities hold*

$$\alpha_{R_1, M_-, M_+, -M_-}(\xi) = \sigma_{R_1, M_-, M_+, 1, 0}(\xi) \alpha_{R_1, M_-, M_+ - 1, -M_-}(\xi) \quad (37a)$$

$$\begin{aligned} \alpha_{R_1, M_-, M_+, \ell}(\xi) & = \sigma_{R_1, M_-, M_+, 1, 0}(\xi) \alpha_{R_1, M_-, M_+ - 1, \ell}(\xi) \\ & \quad + \sigma_{R_1, M_-, M_+, 1, 1}(\xi) \alpha_{R_1, M_-, M_+, \ell}(\xi) \quad \forall \ell \in \{-M_- + 1, \dots, M_+ - 1\} \end{aligned} \quad (37b)$$

$$\alpha_{R_1, M_-, M_+, M_+}(\xi) = \sigma_{R_1, M_-, M_+, 1, 1}(\xi) \alpha_{R_1, M_-, M_+, M_+}(\xi) \quad (37c)$$

$$0 = \sigma_{R_1, M_-, M_+, 1, 0}(\xi) \lambda_{R_1, M_-, M_+ - 1, M}(\xi) + \sigma_{R_1, M_-, M_+, 1, 1}(\xi) \lambda_{R_1, M_-, M_+, M}(\xi) \quad (37d)$$

$$\begin{aligned} \lambda_{R_1, M_-, M_+, n}(\xi) & = \sigma_{R_1, M_-, M_+, 1, 0}(\xi) \lambda_{R_1, M_-, M_+ - 1, n}(\xi) + \sigma_{R_1, M_-, M_+, 1, 1}(\xi) \lambda_{R_1, M_-, M_+, n}(\xi) \quad \forall n \geq M + 1 \\ & \quad \forall \xi \in \mathbb{R} \setminus \{\xi \in \mathbb{R} : \alpha_{R_1, M_-, M_+, M_+}(\xi) = 0\} \end{aligned} \quad (37e)$$

¹⁸A more detailed proof is given in Proposition 4.7.

PROOF. We have already proved (37a) as (34g), (37c) as (34h), and (37b) as (34j). They are summarized separately here for future use. Identity (37d) follows directly from the definitions of $\sigma_{R_1, M_-, M_+, 1, 0}(\xi)$ (32a) and $\sigma_{R_1, M_-, M_+, 1, 1}(\xi)$ (32b), and was used in the calculations leading to (34d). To prove the relation (37e), we replace $p_{R_1, M_-, M_+}(x_i + \xi\Delta x; x_i, \Delta x; f)$, $p_{R_1, M_-, M_+ - 1}(x_i + \xi\Delta x; x_i, \Delta x; f)$ and $p_{R_1, M_- - 1, M_+}(x_i + \xi\Delta x; x_i, \Delta x; f)$ in (32e) by their expansions in terms of the derivatives $h^{(n)}(x_i + \xi\Delta x)$ (8c), and obtain, using (32c, 37d)

$$\sum_{n=M+1}^{N_{\text{tr}}} \left(\sigma_{R_1, M_-, M_+, 1, 0}(\xi) \lambda_{R_1, M_-, M_+ - 1, n}(\xi) + \sigma_{R_1, M_-, M_+, 1, 1}(\xi) \lambda_{R_1, M_- - 1, M_+, n}(\xi) \right) \Delta x^n h^{(n)}(x + \xi\Delta x) \stackrel{(32c, 37d)}{=} O(\Delta x^{N_{\text{tr}}+1}) \quad (38)$$

$$\forall \xi \in \mathbb{R} \setminus \{\xi \in \mathbb{R} : \alpha_{R_1, M_- - 1, M_+, M_+}(\xi) = 0\} \quad \forall h \in C^{N_{\text{tr}}+1}(\mathbb{R}).$$

Using polynomials $q(x) \in \mathbb{R}_n[x]$ (Remark 2.4), recursively for $n \geq M + 1$, in (38) proves (37e), by induction. \square

4.3. ($K_s \geq 1$)-level subdivision

We have shown in [12, Lemma 2.1] that if a general family of functions $p_{M_-, M_+}(x)$, depending on 2 integer indices $M_{\pm} \in \mathbb{Z} : M_- + M_+ \geq 1$, is equipped with a ($K_s = 1$)-level subdivision rule, defined by a relation of the form (5a) with $K_s = 1$, then, by recurrence, we can construct weight-functions satisfying (5a) $\forall K_s \leq M - 1$. By Lemma 4.2, we can always define uniquely the optimal weight-functions (32) of the ($K_s = 1$)-level subdivision of S_{i, M_-, M_+} (Definition 1.2). Therefore, the Lagrange reconstructing polynomials $p_{R_1, M_-, M_+}(x_i + \xi\Delta x; x_i, \Delta x; f)$ (8a) satisfy the conditions of [12, Lemma 2.1].

Proposition 4.5 (Recursive generation of weight-functions for the Lagrange reconstructing polynomial). *Assume the conditions of Lemma 4.2. Then, $\forall M_{\pm} \in \mathbb{Z} : M := M_- + M_+ \geq 2$, $\forall K_s \leq M - 1$, the reconstructing polynomial on S_{i, M_-, M_+} (Proposition 2.2) can be represented, almost everywhere, by combination of the reconstructing polynomials on the K_s -level substencils (Definition 1.2) of S_{i, M_-, M_+} , as*

$$p_{R_1, M_-, M_+}(x_i + \xi\Delta x; x_i, \Delta x; f) = \sum_{k_s=0}^{K_s} \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) p_{R_1, M_- - k_s, M_+ - K_s + k_s}(x_i + \xi\Delta x; x_i, \Delta x; f) \quad (39a)$$

$$\forall \xi \in \mathbb{R} \setminus \mathcal{S}_{R_1, M_-, M_+, K_s} \quad \forall x_i \in \mathbb{R} \quad \forall \Delta x \in \mathbb{R}_{>0} \quad \forall f : \mathbb{R} \rightarrow \mathbb{R}$$

where the rational weight-functions $\sigma_{R_1, M_-, M_+, K_s, k_s}(\xi)$ are defined recursively by

$$\sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) = \begin{cases} \frac{\alpha_{R_1, M_-, M_+, -M_- + k_s M}(\xi)}{\alpha_{R_1, M_- - k_s, M_+ - 1 + k_s, -M_- + k_s M}(\xi)} & K_s = 1 \\ \sum_{\ell_s = \max(0, k_s - 1)}^{\min(K_s - 1, k_s)} \sigma_{R_1, M_-, M_+, K_s - 1, \ell_s}(\xi) \sigma_{R_1, M_- - \ell_s, M_+ - (K_s - 1) + \ell_s, 1, k_s - \ell_s}(\xi) & K_s \geq 2 \end{cases} \quad (39b)$$

$$\forall k_s \in \{0, \dots, K_s\} \quad \forall K_s \in \{1, \dots, M - 1\}$$

and satisfy the consistency condition

$$\sum_{k_s=0}^{K_s} \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) = 1 \quad \forall \xi \in \mathbb{R} \quad (39c)$$

The set of poles of the rational weight-functions $\mathcal{S}_{R_1, M_-, M_+, K_s}$ (39a) satisfies

$$\mathcal{S}_{R_1, M_-, M_+, 1} := \{\xi \in \mathbb{R} : \alpha_{R_1, M_- - 1, M_+, M_+}(\xi) = 0\} \stackrel{(24)}{=} \{\xi_{R_1, M_- - 1, M_+, M_+, n}; n \in \{-M_- + 1, \dots, M_+ - 1\}\} \quad (39d)$$

$$\mathcal{S}_{R_1, M_-, M_+, K_s} \subseteq \bigcup_{L_s=0}^{K_s-1} \bigcup_{\ell_s=0}^{L_s} \mathcal{S}_{\sigma_{M_- - \ell_s, M_+ - L_s + \ell_s, 1}} = \left\{ \xi \in \mathbb{R} : \prod_{L_s=0}^{K_s-1} \prod_{\ell_s=0}^{L_s} \alpha_{R_1, M_- - 1 - \ell_s, M_+ - L_s + \ell_s, M_+ - L_s + \ell_s}(\xi) = 0 \right\} \quad (39e)$$

$$\forall K_s \in \{1, \dots, M - 1\}$$

PROOF. The case $K_s = 1$ follows from Lemma 4.2, with the set of isolated singular points $\mathcal{S}_{R_1, M_-, M_+, 1}$ defined by (39d), because of (32e). Since the $(K_s = 1)$ -level subdivision rule is established, the conditions of [12, Lemma 2.1] are satisfied, proving (39a–39c), and the recursive definition (39e) of the set of isolated singular points $\mathcal{S}_{R_1, M_-, M_+, K_s}$. The \subseteq relation is used in (39e) for $K_s > 1$, because there may be pole cancellation by the multiplications in (39b). \square

Corollary 4.6 (Representation of the fundamental polynomials $\alpha_{R_1, M_-, M_+, \ell}(\xi)$ (11a)). *Assume the conditions of Proposition 4.5. Then, the fundamental polynomials of Lagrange reconstruction $\alpha_{R_1, M_-, M_+, \ell}(\xi)$ (11a) on S_{i, M_-, M_+} (Definition 1.1), can be represented by a weighted combination of the basis (Proposition 3.3) Lagrange reconstructing polynomials on the K_s -level substencils (Definition 1.2) of S_{i, M_-, M_+} as*

$$\alpha_{R_1, M_-, M_+, \ell}(\xi) = \sum_{k_s=\max(0, \ell+K_s-M_+)}^{\min(K_s, \ell+M_-)} \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) \alpha_{R_1, M_-, M_+, -k_s, M_+-K_s+k_s, \ell}(\xi) \quad \forall \xi \in \mathbb{R} \setminus \mathcal{S}_{R_1, M_-, M_+, K_s} \quad (40)$$

where the weight-functions $\sigma_{R_1, M_-, M_+, K_s, k_s}(\xi)$ are defined by (39b) in Proposition 4.5, and the set of isolated singular points $\mathcal{S}_{R_1, M_-, M_+, K_s}$ by (39d, 39e).

PROOF. Rewrite (39a) as¹⁹

$$\begin{aligned} p_{R_1, M_-, M_+}(x_i + \xi \Delta x; x_i, \Delta x; f) &\stackrel{(39a)}{=} \sum_{k_s=0}^{K_s} \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) p_{R_1, M_-, M_+, -k_s, M_+-K_s+k_s}(x_i + \xi \Delta x; x_i, \Delta x; f) \\ &\stackrel{(8a)}{=} \sum_{k_s=0}^{K_s} \left(\sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) \sum_{\ell=-M_++k_s}^{M_+-K_s+k_s} \left(\alpha_{R_1, M_-, M_+, -k_s, M_+-K_s+k_s, \ell}(\xi) f(x_i + \ell \Delta x) \right) \right) \\ &\stackrel{(fn19)}{=} \sum_{\ell=-M_-}^{M_+} \underbrace{\left(\sum_{k_s=\max(0, \ell+K_s-M_+)}^{\min(K_s, \ell+M_-)} \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) \alpha_{R_1, M_-, M_+, -k_s, M_+-K_s+k_s, \ell}(\xi) \right)}_{\alpha_{R_1, M_-, M_+, \ell}(\xi)} f(x_i + \ell \Delta x) \quad (41) \end{aligned}$$

proving (40) by (8a). \square

Proposition 4.7 (Uniqueness of weight-functions $\sigma_{R_1, M_-, M_+, K_s, k_s}(\xi)$ (39b)). *Assume the conditions of Proposition 4.5. The functions $\sigma_{R_1, M_-, M_+, K_s, k_s}(\xi)$ satisfying (39a) are unique.*

PROOF. We have proved by construction (39b) the existence of weight-functions $\sigma_{R_1, M_-, M_+, K_s, k_s}(\xi)$ satisfying (39a, 39c). Uniqueness for the case $K_s = 1$ was proved in Lemma 4.2. Notice first that Corollary 4.6 does not require the validity of the particular expression (39b) of the weight-functions, and is therefore valid for any set of weight-functions $\sigma_{R_1, M_-, M_+, K_s, k_s}(\xi)$ satisfying (39a). To prove therefore uniqueness we can use (40), which can be explicitly

¹⁹

$$\begin{array}{ccccc} 0 \leq k_s \leq K_s & \stackrel{(2b)}{\iff} & 0 \leq k_s \leq K_s & \iff & 0 \leq k_s \leq K_s \\ -M_- + k_s \leq \ell \leq M_+ - K_s + k_s & & -M_- \leq \ell \leq M_+ & & -M_- \leq \ell \leq M_+ \\ & & -M_- + k_s \leq \ell \leq M_+ - K_s + k_s & \iff & k_s \leq \ell + M_- \\ & & & & \ell - M_+ + K_s \leq k_s \end{array}$$

written as

$$\forall K_s \geq 1 \quad \alpha_{R_1, M_-, M_+, -M_-}(\xi) = \sigma_{R_1, M_-, M_+, K_s, 0}(\xi) \alpha_{R_1, M_-, M_+, -K_s, -M_-}(\xi) \quad (42a)$$

$$\begin{aligned} \forall K_s \geq 1 \quad \alpha_{R_1, M_-, M_+, -M_-+1}(\xi) = & \sigma_{R_1, M_-, M_+, K_s, 0}(\xi) \alpha_{R_1, M_-, M_+, -K_s, -M_-+1}(\xi) \\ & + \sigma_{R_1, M_-, M_+, K_s, 1}(\xi) \alpha_{R_1, M_-, M_+, -K_s+1, -M_-+1}(\xi) \end{aligned} \quad (42b)$$

$$\begin{aligned} \forall K_s \geq 2 \quad \alpha_{R_1, M_-, M_+, -M_-+2}(\xi) = & \sigma_{R_1, M_-, M_+, K_s, 0}(\xi) \alpha_{R_1, M_-, M_+, -K_s, -M_-+2}(\xi) \\ & + \sigma_{R_1, M_-, M_+, K_s, 1}(\xi) \alpha_{R_1, M_-, M_+, -K_s+1, -M_-+2}(\xi) \\ & + \sigma_{R_1, M_-, M_+, K_s, 2}(\xi) \alpha_{R_1, M_-, M_+, -K_s+2, -M_-+2}(\xi) \end{aligned} \quad (42c)$$

\vdots

$$\forall \ell \in \{K_s - M_-, M_+ - K_s\} \quad \alpha_{R_1, M_-, M_+, \ell}(\xi) = \sum_{k_s = \max(0, \ell + K_s - M_+)}^{\min(K_s, \ell + M_-)} \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) \alpha_{R_1, M_-, M_+, -K_s + k_s, \ell}(\xi) \quad (42d)$$

\vdots

$$\begin{aligned} \forall K_s \geq 2 \quad \alpha_{R_1, M_-, M_+, M_+-1}(\xi) = & \sigma_{R_1, M_-, M_+, K_s, K_s-2}(\xi) \alpha_{R_1, M_-, M_+, -K_s+2, M_+-2, M_+-2}(\xi) \\ & + \sigma_{R_1, M_-, M_+, K_s, K_s-1}(\xi) \alpha_{R_1, M_-, M_+, -K_s+1, M_+-1, M_+-2}(\xi) \\ & + \sigma_{R_1, M_-, M_+, K_s, K_s}(\xi) \alpha_{R_1, M_-, M_+, -K_s, M_+, M_+-2}(\xi) \end{aligned} \quad (42e)$$

$$\begin{aligned} \forall K_s \geq 1 \quad \alpha_{R_1, M_-, M_+, M_+}(\xi) = & \sigma_{R_1, M_-, M_+, K_s, K_s-1}(\xi) \alpha_{R_1, M_-, M_+, -K_s+1, M_+-1, M_+-1}(\xi) \\ & + \sigma_{R_1, M_-, M_+, K_s, K_s}(\xi) \alpha_{R_1, M_-, M_+, -K_s, M_+, M_+-1}(\xi) \end{aligned} \quad (42f)$$

$$\forall K_s \geq 1 \quad \alpha_{R_1, M_-, M_+, M_+}(\xi) = \sigma_{R_1, M_-, M_+, K_s, K_s}(\xi) \alpha_{R_1, M_-, M_+, -K_s, M_+, M_+}(\xi) \quad (42g)$$

Starting with (42a, 42g) we immediately prove uniqueness of $\sigma_{R_1, M_-, M_+, K_s, 0}(\xi)$ and $\sigma_{R_1, M_-, M_+, K_s, K_s}(\xi)$, by contradiction because of (19b). Having proved uniqueness of $\sigma_{R_1, M_-, M_+, K_s, 0}(\xi)$, (42b) proves uniqueness of $\sigma_{R_1, M_-, M_+, K_s, 1}(\xi)$, by contradiction because of (19b). In exactly the same way, having proved uniqueness of $\sigma_{R_1, M_-, M_+, K_s, K_s}(\xi)$, (42f) proves uniqueness of $\sigma_{R_1, M_-, M_+, K_s, K_s-1}(\xi)$. Continuing the procedure until reaching $\sigma_{R_1, M_-, M_+, K_s, \lceil \frac{K_s}{2} \rceil}(\xi)$ (for increasing k_s , starting from (42a)) and $\sigma_{R_1, M_-, M_+, K_s, \lfloor \frac{K_s}{2} \rfloor}(\xi)$ (for decreasing k_s , starting from (42g)), completes the proof of uniqueness. \square

Corollary 4.8 (Weight-functions and approximation-errors). *Assume the conditions of Proposition 4.5. Then, provided that the reconstruction pair (Definition 1.3) of $f(x)$, $h(x) := [R_{(1;\Delta x)}(f)](x)$, is sufficiently smooth $\forall x \in [x_{i-M_-} - \frac{1}{2}\Delta x, x_{i+M_+} + \frac{1}{2}\Delta x]$, for the expansions (8b, 8c) of the approximation error to hold, the weight-functions $\sigma_{R_1, M_-, M_+, K_s, k_s}(\xi)$ (39b) and the approximation-error polynomials $\mu_{R_1, M_-, M_+, n}(\xi)$ (11b) and $\lambda_{R_1, M_-, M_+, n}(\xi)$ (11c), satisfy*

$$\sum_{k_s=0}^{K_s} \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) \lambda_{R_1, M_-, M_+, -K_s+k_s, n}(\xi) = 0 \quad \forall n \in \{M - K_s + 1, M\} \quad (43a)$$

$$\sum_{k_s=0}^{K_s} \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) \lambda_{R_1, M_-, M_+, -K_s+k_s, n}(\xi) = \lambda_{R_1, M_-, M_+, n}(\xi) \quad \forall n \geq M + 1 \quad (43b)$$

$$\sum_{k_s=0}^{K_s} \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) \mu_{R_1, M_-, M_+, -K_s+k_s, n}(\xi) = 0 \quad \forall n \in \{M - K_s + 1, M\} \quad (43c)$$

$$\sum_{k_s=0}^{K_s} \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) \mu_{R_1, M_-, M_+, -K_s+k_s, n}(\xi) = \mu_{R_1, M_-, M_+, n}(\xi) \quad \forall n \geq M + 1 \quad (43d)$$

$$\forall \xi \in \mathbb{R} \setminus \mathcal{S}_{R_1, M_-, M_+, K_s}$$

where the set of isolated singular points $\mathcal{S}_{R_1, M_-, M_+, K_s}$ is defined by (39d, 39e).

PROOF. The proof is quite obvious by replacing $p_{R_1, M_-, M_+}(x_i + \xi \Delta x; x_i, \Delta x; f)$ and $p_{R_1, M_-, M_+ - K_s + k_s}(x_i + \xi \Delta x; x_i, \Delta x; f)$ in (39a) by either (8b) or (8c), yielding

$$\begin{aligned}
p_{R_1, M_-, M_+}(x_i + \xi \Delta x; x_i, \Delta x; f) &\stackrel{(8c)}{=} h(x_i + \xi \Delta x) + \sum_{n=M+1}^{N_{TJ}} \lambda_{R_1, M_-, M_+, n}(\xi) \Delta x^n h^{(n)}(x_i + \xi \Delta x) + O(\Delta x^{N_{TJ}+1}) \\
&\stackrel{(39a)}{=} \sum_{k_s=0}^{K_s} \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) p_{R_1, M_-, M_+ - K_s + k_s}(x_i + \xi \Delta x; x_i, \Delta x; f) \\
&\stackrel{(8c)}{=} \underbrace{\left(\sum_{k_s=0}^{K_s} \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) \right)}_{=1 \text{ (39c)}} h(x_i + \xi \Delta x) \\
&\quad + \sum_{n=M-K_s+1}^{N_{TJ}} \left(\sum_{k_s=0}^{K_s} \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) \lambda_{R_1, M_-, M_+ - K_s + k_s, n}(\xi) \right) \Delta x^n h^{(n)}(x_i + \xi \Delta x) \\
&\quad + O(\Delta x^{N_{TJ}+1}) \tag{44a}
\end{aligned}$$

$$\begin{aligned}
p_{R_1, M_-, M_+}(x_i + \xi \Delta x; x_i, \Delta x; f) &\stackrel{(8b)}{=} h(x_i + \xi \Delta x) + \sum_{n=M+1}^{N_{TJ}} \mu_{R_1, M_-, M_+, n}(\xi) \Delta x^n f^{(n)}(x_i) + O(\Delta x^{N_{TJ}+1}) \\
&\stackrel{(39a)}{=} \sum_{k_s=0}^{K_s} \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) p_{R_1, M_-, M_+ - K_s + k_s}(x_i + \xi \Delta x; x_i, \Delta x; f) \\
&\stackrel{(8b)}{=} \underbrace{\left(\sum_{k_s=0}^{K_s} \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) \right)}_{=1 \text{ (39c)}} h(x_i + \xi \Delta x) \\
&\quad + \sum_{n=M-K_s+1}^{N_{TJ}} \left(\sum_{k_s=0}^{K_s} \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) \mu_{R_1, M_-, M_+ - K_s + k_s, n}(\xi) \right) \Delta x^n f^{(n)}(x_i) \\
&\quad + O(\Delta x^{N_{TJ}+1}) \tag{44b}
\end{aligned}$$

$$\forall \xi \in \mathbb{R} \setminus \mathcal{S}_{R_1, M_-, M_+, K_s} \quad \forall x_i \in \mathbb{R} \quad \forall \Delta x \in \mathbb{R}_{>0} \quad \forall h \in C^{N_{TJ}+1}(\mathbb{R}) \quad f := R_{(1, \Delta x)}^{-1}(h)$$

which prove (43) by identification of coefficients of Δx^n . \square

Example 4.9 (Rational weight-functions $\sigma_{R_1, M_-, M_+, K_s, k_s}(\xi)$ (39b)). The stencil $s_{i,3,3}$ ($M_- = 3$, $M_+ = 3$, $M := M_- + M_+ = 6$) is symmetric around $\xi = 0$ (Fig. 3). The ($K_s = \lceil \frac{M}{2} \rceil = 3$)-level subdivision (Definition 1.2) is the highest level of subdivision for which all of the substencils contain either point i or point $i + 1$ (Fig. 3). The rational weight-functions $\sigma_{R_1, 3, 3, 3, k_s}(\xi)$ ($k_s \in \{0, \dots, 3\}$) are all > 0 in the interval $I_{C_{R_1}(\frac{1}{2}), 3, 3, 3}$ around point $\xi = +\frac{1}{2}$ (Fig. 5). Because of the symmetry of the stencil $s_{i,3,3}$ around $\xi = 0$ (Fig. 5), we also have $\sigma_{R_1, 3, 3, 3, k_s}(-\frac{1}{2}) > 0 \quad \forall k_s \in \{0, \dots, 3\}$. The stencil $s_{i,3,4}$ ($M_- = 3$, $M_+ = 4$, $M := M_- + M_+ = 7$) is symmetric around $\xi = \frac{1}{2}$ (Fig. 4). The ($K_s = \lceil \frac{M}{2} \rceil = 4$)-level subdivision (Definition 1.2) is the highest level of subdivision for which all of the substencils contain either point i or point $i + 1$ (Fig. 4). The rational weight-functions $\sigma_{R_1, 3, 4, 4, k_s}(\xi)$ ($k_s \in \{0, \dots, 4\}$) are all > 0 in the interval $I_{C_{R_1}(\frac{1}{2}), 3, 4, 4}$ around point $\xi = +\frac{1}{2}$ (Fig. 6). The stencil $s_{i,3,4}$ not being symmetric around $\xi = 0$ (Fig. 6), positivity of the weight-functions does not hold around $\xi = -\frac{1}{2}$, where $\sigma_{R_1, 3, 4, 4, 4}(-\frac{1}{2}) = -\frac{3}{770}$, by direct computation using (39b). The conditions of positivity of weight-functions at $\xi = +\frac{1}{2}$, which is important in the development of WENO schemes [3], are studied below (§4.4). \square

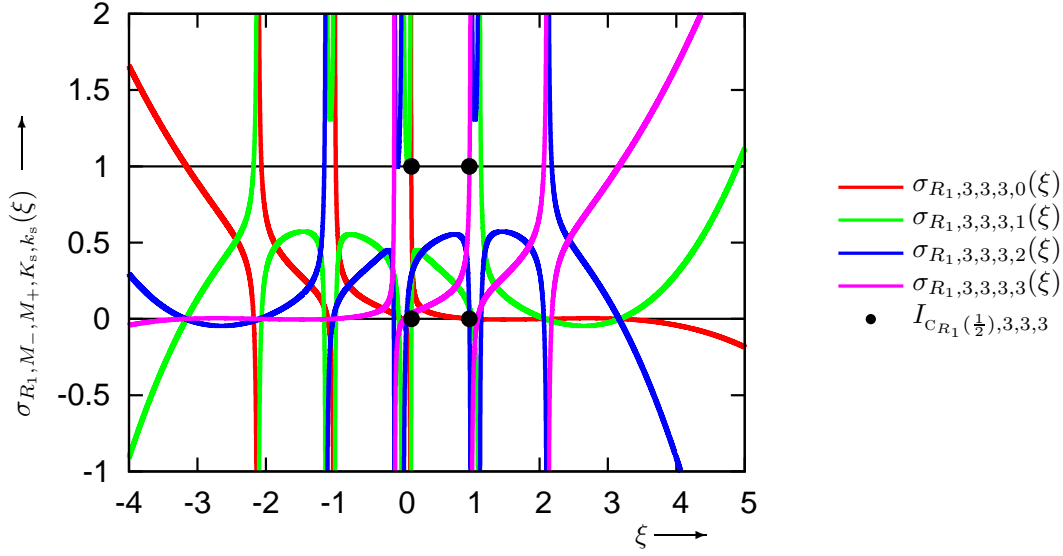


Figure 5: Rational weight-functions $\sigma_{R_1, M_-, M_+, K_s, k_s}(\xi)$ (39b) for the $(K_s = 3)$ -level subdivision (Definition 1.2) of the stencil $S_{i,3,3}$ (Fig. 1), and interval of convexity of the weight-functions around $i + \frac{1}{2}$, $I_{C_{R_1}(\frac{1}{2}), M_-, M_+, K_s}$ (Theorem 4.14).

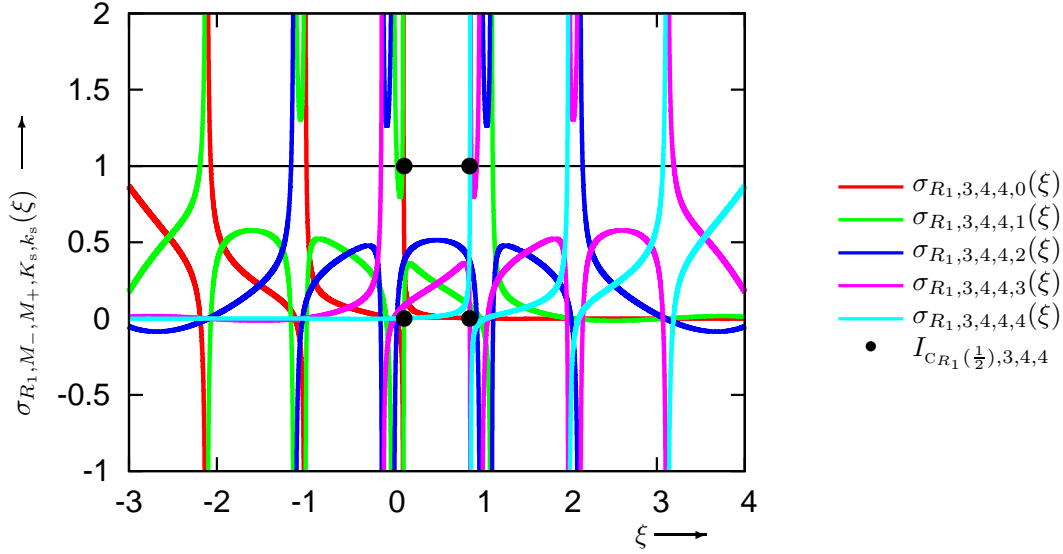


Figure 6: Rational weight-functions $\sigma_{R_1, M_-, M_+, K_s, k_s}(\xi)$ (39b) for the $(K_s = 4)$ -level subdivision (Definition 1.2) of the stencil $S_{i,3,4}$ (Fig. 2), and interval of convexity of the weight-functions around $i + \frac{1}{2}$, $I_{C_{R_1}(\frac{1}{2}), M_-, M_+, K_s}$ (Theorem 4.14).

4.4. Convexity

The nonlinear modification of the optimal (linear) weights in WENO schemes [2, 9] is more straightforward when the combination (5a) is convex [20].

Remark 4.10 (Consistency, positivity and convexity). As can be seen by (44a, 44b), condition (39c) ensures the consistency of the representation (39a) as an approximation of $h(x) =: [R_{(1;\Delta x)}(f)](x)$ (Definition 1.3), and is therefore called the consistency condition of the representation (39a). Obviously, when at a fixed $\xi \in \mathbb{R}$ all of the K_s -level-subdivision weight-functions are ≥ 0 then, because of (39c), they must take values $\in [0, 1]$ (proof by contradiction)

$$\left[\sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) \geq 0 \quad \forall k_s \in \{0, \dots, K_s\} \right] \stackrel{(39c)}{\iff} \left[0 \leq \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) \leq 1 \quad \forall k_s \in \{0, \dots, K_s\} \right] \quad (45)$$

Hence, positivity of the weight-functions at a fixed $\xi \in \mathbb{R}$ ensures, by the consistency condition (39c), that, locally, the representation (39a) is convex. \square

In the early WENO papers [1, 2] convexity of the combination (5a) had been postulated, and verified by direct determination of the coefficients at $\xi = \frac{1}{2}$ [2, 9]. Shu [5] showed examples of combinations of choices of the stencil S_{i,M_-,M_+} (Definition 1.1), of the level of subdivision K_s (Definition 1.2), and of the location $\xi \in \mathbb{R}$, for which convexity of (5a) is lost, and this appeared as a practical problem, not only in 2-D and 3-D unstructured grids [20], but also in the development of centered (central) WENO schemes [15]. For this reason the intervals of convexity were investigated numerically [5, 3, 11].

The analytical results obtained in the present work, in particular the recursive analytical expression of the weight-functions $\sigma_{R_1,M_-,M_+,K_s,k_s}(\xi)$ (Proposition 4.5) and the factorization of the fundamental functions of Lagrange reconstruction $\alpha_{R_1,M_-,M_+,l}(\xi)$ (Proposition 3.7), can be used to study convexity intervals for arbitrary values of $[M_\pm, K_s]$, as was recently done for the Lagrange interpolating polynomial [12, Proposition 3.2]. In [7, Result 6.1, p. 300] we had conjectured that for any choice of $[M_\pm, K_s]$ for which all of the substencils $S_{i,M_- - k_s, M_+ - K_s + k_s}$ ($k_s \in \{0, \dots, K_s\}$) contain either point i or point $i + 1$ (or both), convexity was observed at $\xi = \frac{1}{2}$. We provide here a formal proof of this conjecture, and give an estimate of the interval of convexity around $\xi = \frac{1}{2}$.

Lemma 4.11 (Positive subdivision). *Consider the subdivision level $K_s \geq 1$ of S_{i,M_-,M_+} (Definition 1.2). Iff*

$$-M_- \leq 0 < 1 \leq M_+ \quad (46a)$$

$$1 \leq K_s \leq \min(M_- + 1, M_+) \quad (46b)$$

then all substencils contain either point i or point $i + 1$

$$S_{i,M_- - k_s, M_+ - K_s + k_s} \cap \{i, i + 1\} \neq \emptyset \quad \forall k_s \in \{0, \dots, K_s\} \quad (46c)$$

More precisely

$$(46a, 46b) \iff (46c) \iff \left[\begin{array}{ll} \{i, i + 1\} & \subseteq S_{i,M_-,M_+} \\ i & \in S_{i,M_-,M_+ - K_s} \\ \{i, i + 1\} & \subseteq S_{i,M_- - k_s, M_+ - K_s + k_s} \quad \forall k_s \in \{1, \dots, K_s - 1\} \\ i + 1 & \in S_{i,M_- - K_s, M_+} \end{array} \right] \quad (46d)$$

A subdivision (Definition 1.2) satisfying (46d) will be called a positive subdivision [7, Result 6.1, p. 300].

PROOF. First notice that if all substencils contain either point i or point $i + 1$ (46c) then so does the entire stencil $S_{i,M_-,M_+} \stackrel{(2d)}{=} \bigcup_{k_s=0}^{K_s} S_{i,M_- - k_s, M_+ - K_s + k_s}$. Taking into account that by hypothesis $K_s \geq 1$, in the condition $\{i, i + 1\} \cap S_{i,M_-,M_+} \neq \emptyset$, implies that S_{i,M_-,M_+} must contain both points i and $i + 1$ (proof²⁰ by contradiction taking into account $K_s \geq 1$). The condition that both points $\{i, i + 1\}$ must be contained in the big stencil S_{i,M_-,M_+} yields

$$(46c) \stackrel{(2d)}{\implies} \{i, i + 1\} \cap S_{i,M_-,M_+} \neq \emptyset \stackrel{(fn20)}{\implies} \{i, i + 1\} \subset \{i - M_-, \dots, i + M_+\} \iff i - M_- \leq i < i + 1 \leq i + M_+ \iff -M_- \leq 0 < 1 \leq M_+ \quad (47a)$$

proving that (46a) is a necessary condition for the validity of (46c). Combining (46c, 47a) implies (proof²¹ by contradiction) that i must belong to the leftmost substencil ($k_s = 0$) and $i + 1$ must belong to the rightmost substencil

²⁰Since by (46c) each of the substencils $S_{i,M_- - k_s, M_+ - K_s + k_s}$ ($k_s \in \{0, \dots, K_s \geq 1\}$) has a non-empty intersection with $\{i, i + 1\}$, so does their union S_{i,M_-,M_+} (2d), ie $\{i, i + 1\} \cap S_{i,M_-,M_+} \neq \emptyset$. Obviously the conditions $((i + M_+ < i < i + 1) \vee (i < i + 1 < i - M_-)) \implies \{i, i + 1\} \cap S_{i,M_-,M_+} = \emptyset$ are a contradiction, implying that their negation is true, ie we must have $((i + M_+ \geq i) \wedge (i + 1 \geq i - M_-))$. It turns out that the inequalities in $((i + M_+ \geq i) \wedge (i + 1 \geq i - M_-))$ must be strict. Assuming $i + 1 = i - M_- \implies i < i + 1 < i - M_- + 1 \implies \{i, i + 1\} \cap S_{i,M_- - 1, M_+ - K_s + 1} = \emptyset$ contradicts (46c), implying $i + 1 > i - M_- \implies M_- > -1 \stackrel{M_- \in \mathbb{Z}}{\implies} M_- \geq 0$. Assuming $i = i + M_+ \implies i + 1 > i > i + M_+ - K_s + (K_s - 1) \implies \{i, i + 1\} \cap S_{i,M_- + (K_s - 1), M_+ - K_s + (K_s - 1)} = \emptyset$ contradicts (46c), because by hypothesis $K_s \geq 1$, implying $i < i + M_+ \implies M_+ > 0 \stackrel{M_+ \in \mathbb{Z}}{\implies} M_+ \geq 1$.

²¹Assuming $i \notin S_{i,M_- - K_s, M_+} \stackrel{(47a)}{\implies} i > i + M_+ - K_s \implies \{i, i + 1\} \cap S_{i,M_- - K_s, M_+} = \emptyset$ contradicts (46c) for $k_s = 0$. Assuming $i + 1 \notin S_{i,M_- + K_s, M_+} \stackrel{(47a)}{\implies} i + 1 < i - M_- + K_s \implies \{i, i + 1\} \cap S_{i,M_- + K_s, M_+} = \emptyset$ contradicts (46c) for $k_s = K_s \geq 1$.

$(k_s = K_s \geq 1)$

$$(46c, 47a) \xRightarrow{(fn21)} \left[\begin{array}{l} i \in S_{i,M_-,M_+-K_s} \implies i - M_- \leq i \leq i + M_+ - K_s \\ i + 1 \in S_{i,M_-+K_s,M_+} \implies i - M_- + K_s \leq i + 1 \leq i + M_+ \end{array} \right] \implies K_s \leq \min(M_- + 1, M_+) \quad (47b)$$

proving that (46b) is also a necessary condition for the validity of (46c). To complete the proof it suffices to show that (46a, 46b) are not only necessary but also sufficient conditions for (46c). We have

$$\begin{aligned} (46a, 46b) &\implies \left[\begin{array}{l} M_+ \geq 1 \\ -M_- \leq 0 \\ -M_- + K_s \leq 1 \\ M_+ - K_s \geq 0 \end{array} \right] \implies \left[\begin{array}{l} i - M_- \leq i < i + 1 \leq i + M_+ \\ i - M_- + K_s \leq i + 1 \leq i + M_+ - K_s \end{array} \right] \\ &\implies \left[\begin{array}{l} i - M_- \leq i < i + 1 \leq i + M_+ \\ i - M_- \leq i \leq i + M_+ - K_s \\ i - M_- + k_s < i + 1 \leq i + M_+ - K_s + k_s \quad \forall k_s \in \{0, \dots, K_s - 1\} \\ i - M_- + K_s \leq i + 1 \leq i + M_+ \quad \forall k_s \in \{1, \dots, K_s\} \end{array} \right] \\ &\implies \left[\begin{array}{l} i - M_- \leq i < i + 1 \leq i + M_+ \\ i - M_- \leq i \leq i + M_+ - K_s \\ i - M_- + k_s \leq i < i + 1 \leq i + M_+ - K_s + k_s \quad \forall k_s \in \{1, \dots, K_s - 1\} \\ i - M_- + K_s \leq i + 1 \leq i + M_+ \end{array} \right] \quad (47c) \end{aligned}$$

completing the proof, the last conditions in (47c) being exactly (46d). \square

Corollary 4.12 ($(K_s = 1)$ -level positively subdivisible stencils). *Assume that $M_{\pm} \in \mathbb{Z} : M := M_- + M_+ \geq 2$ defining the stencil S_{i,M_-,M_+} (Definition 1.1) satisfy $-M_- \leq 0 < 1 \leq M_+$ (46a). Then the $(K_s = 1)$ -level subdivision of the stencil S_{i,M_-,M_+} (Definition 1.2) is a positive subdivision (Lemma 4.11).*

PROOF. By (46a) we have that $((M_- \geq 0) \wedge (M_+ \geq 1)) \implies \min(M_- + 1, M_+) \geq 1$, so that $K_s = 1 \leq \min(M_- + 1, M_+)$. Hence the conditions (46a, 46b) are satisfied, so that, by Lemma 4.11, the $(K_s = 1)$ -level subdivision of a stencil satisfying (46a) is a positive subdivision. \square

Lemma 4.13 (Convexity in the neighborhood of $i + \frac{1}{2}$ for $(K_s = 1)$ -level subdivision). *Assume that $M_{\pm} \in \mathbb{Z} : M := M_- + M_+ \geq 2$ (2a) defining the stencil S_{i,M_-,M_+} (Definition 1.1) satisfy $-M_- \leq 0 < 1 \leq M_+$ (46a). Then the rational weight-functions $\sigma_{R_1,M_-,M_+,1,0}(\xi)$ (32a) and $\sigma_{R_1,M_-,M_+,1,1}(\xi)$ (32b) for the representation of the Lagrange reconstructing polynomial by the Lagrange reconstructing polynomials of the $(K_s = 1)$ -level substencils of S_{i,M_-,M_+} (Lemma 4.2) satisfy*

$$0 < \sigma_{R_1,M_-,M_+,1,k_s}(\xi) < 1 \quad \left\{ \begin{array}{l} \forall \xi \in I_{C_{R_1}(\frac{1}{2}),M_-,M_+,1} := \left(\xi_{C_{R_1}(\frac{1}{2}),M_-,M_+,1}^-, \xi_{C_{R_1}(\frac{1}{2}),M_-,M_+,1}^+ \right) \subset \mathbb{R} \\ \forall k_s \in \{0, 1\} \end{array} \right. \quad (48a)$$

where the limits of the convexity interval around $\xi = \frac{1}{2}$, $I_{C_{R_1}(\frac{1}{2}),M_-,M_+,1} \ni \frac{1}{2}$ of length > 0 , are defined by

$$\xi_{C_{R_1}(\frac{1}{2}),M_-,M_+,1}^- := \begin{cases} \xi_{R_1,M_-,M_+,M_+,0} & M_- = 0 \\ \max(\xi_{R_1,M_-,M_+,M_+,0}, \xi_{R_1,M_-,M_+,-M_-,0}, \xi_{R_1,M_-,M_+-1,-M_-,0}) & M_- > 0 \end{cases} \quad (48b)$$

$$\xi_{C_{R_1}(\frac{1}{2}),M_-,M_+,1}^+ := \begin{cases} \xi_{R_1,M_-,M_+,-M_-,1} & M_+ = 1 \\ \min(\xi_{R_1,M_-,M_+,-M_-,1}, \xi_{R_1,M_-,M_+,M_+,1}, \xi_{R_1,M_--1,M_+,M_+,1}) & M_+ > 1 \end{cases} \quad (48c)$$

where $\xi_{R_1,M_-,M_+, \ell, n}$ ($n \in \{-M_-, \dots, M_+\} \setminus \{\ell\}$) are the M real roots (Proposition 3.5) of the fundamental polynomial of Lagrange reconstruction $\alpha_{R_1,M_-,M_+, \ell}(\xi)$ (11a).

PROOF. By hypothesis, the stencil S_{i,M_-,M_+} satisfies the conditions of Corollary 4.12, implying that the $(K_s = 1)$ -level subdivision of S_{i,M_-,M_+} is a positive subdivision, satisfying the conditions of Lemma 4.11, and we have by (46d)

$$\{i, i+1\} \subseteq S_{i,M_-,M_+} \quad (49a)$$

$$i \in S_{i,M_-,M_+-1} \quad (49b)$$

$$i+1 \in S_{i,M_--1,M_+} \quad (49c)$$

By Lemma 4.2 the rational weight-functions $\sigma_{R_1,M_-,M_+,1,0}(\xi)$ (32a) and $\sigma_{R_1,M_-,M_+,1,1}(\xi)$ (32b) can be expressed in terms of the fundamental polynomials of Lagrange reconstruction (Proposition 2.2) $\alpha_{R_1,M_-,M_+,-M_-}(\xi)$, $\alpha_{R_1,M_-,M_+,+M_+}(\xi)$, $\alpha_{R_1,M_-,M_+-1,-M_-}(\xi)$, and $\alpha_{R_1,M_--1,M_+,+M_+}(\xi)$. Notice that, because of the identities of Proposition 3.10, we have $\alpha_{R_1,M_--1,M_+,+M_+}(\xi) \stackrel{(27a)}{=} (-1)^{M_-1} \alpha_{R_1,M_-,M_+-1,-M_-}(\xi)$. By Proposition 3.5, all of the roots of the fundamental polynomials of Lagrange reconstruction are real, and therefore the factorization of Proposition 3.7 applies. Applying the factorization (25a), and taking into account (49a, 49b), which were shown in Lemma 4.11 to be direct consequences of (46a), we have

$$\begin{aligned} \alpha_{R_1,M_-,M_+,-M_-}(\xi) &\stackrel{(25a)}{=} \frac{(-1)^M}{M!} \prod_{n=-M_-+1}^{M_+} (\xi - \xi_{R_1,M_-,M_+,-M_-,n}) \\ &\stackrel{(46a)}{=} \frac{(-1)^M}{M!} \left\{ \begin{array}{ll} \prod_{n=1}^{M_+} (\xi - \xi_{R_1,M_-,M_+,-M_-,n}) & M_- = 0 \\ \prod_{n=-M_-+1}^0 (\xi - \xi_{R_1,M_-,M_+,-M_-,n}) \prod_{n=1}^{M_+} (\xi - \xi_{R_1,M_-,M_+,-M_-,n}) & M_- > 0 \end{array} \right. \end{aligned} \quad (49d)$$

$$\begin{aligned} \alpha_{R_1,M_-,M_+,+M_+}(\xi) &\stackrel{(25a)}{=} \frac{(-1)^{2M_+}}{M!} \prod_{n=-M_-}^{M_+-1} (\xi - \xi_{R_1,M_-,M_+,+M_+,n}) \\ &\stackrel{(46a)}{=} \frac{(-1)^{2M_+}}{M!} \left\{ \begin{array}{ll} \prod_{n=-M_-}^0 (\xi - \xi_{R_1,M_-,M_+,+M_+,n}) & M_+ = 1 \\ \prod_{n=-M_-}^0 (\xi - \xi_{R_1,M_-,M_+,+M_+,n}) \prod_{n=1}^{M_+-1} (\xi - \xi_{R_1,M_-,M_+,+M_+,n}) & M_+ > 1 \end{array} \right. \end{aligned} \quad (49e)$$

$$\begin{aligned} \alpha_{R_1,M_-,M_+-1,-M_-}(\xi) &\stackrel{(25a)}{=} \frac{(-1)^{M-1}}{(M-1)!} \prod_{n=-M_-+1}^{M_+-1} (\xi - \xi_{R_1,M_-,M_+-1,-M_-,n}) \\ &\stackrel{(46a)}{=} \frac{(-1)^{M-1}}{(M-1)!} \left\{ \begin{array}{ll} \prod_{n=1}^{M_+-1} (\xi - \xi_{R_1,M_-,M_+-1,-M_-,n}) & M_- = 0 \\ \prod_{n=-M_-+1}^0 (\xi - \xi_{R_1,M_-,M_+-1,-M_-,n}) \prod_{n=1}^{M_+-1} (\xi - \xi_{R_1,M_-,M_+-1,-M_-,n}) & M_- = 1 \\ \prod_{n=-M_-+1}^0 (\xi - \xi_{R_1,M_-,M_+-1,-M_-,n}) \prod_{n=1}^{M_+-1} (\xi - \xi_{R_1,M_-,M_+-1,-M_-,n}) & M_- \neq 0 \neq M_+ - 1 \end{array} \right. \end{aligned} \quad (49f)$$

where in (49f) we only need to distinguish 3 cases because the constraint $M := M_- + M_+ \geq 2$ (2a) implies that we cannot have simultaneously $M_- = 0$ and $M_+ = 1$. Since by Proposition 3.10, $\alpha_{R_1,M_--1,M_+,+M_+}(\xi) \stackrel{(27a)}{=} (-1)^{M_-1} \alpha_{R_1,M_-,M_+-1,-M_-}(\xi) \implies \xi_{R_1,M_--1,M_+,+M_+,n} = \xi_{R_1,M_-,M_+-1,-M_-,n} \forall n \in \{-M_-+1, \dots, M_+-1\}$, defining the limits

of the open convexity interval around $\xi = \frac{1}{2}$, $I_{C_{R_1}(\frac{1}{2}), M_-, M_+, 1}$ (48a), by (48b, 48c), we have

$$\text{sign}(\alpha_{R_1, M_-, M_+, -M_-}(\xi)) \stackrel{(49d)}{=} (-1)^{M_+} \stackrel{(2a)}{=} (-1)^{M_-} \quad \forall \xi \in I_{C_{R_1}(\frac{1}{2}), M_-, M_+, 1} \quad (49g)$$

$$\text{sign}(\alpha_{R_1, M_-, M_+, +M_+}(\xi)) \stackrel{(49e)}{=} (-1)^{2M_+ + M_+ - 1} = (-1)^{M_+ - 1} \quad \forall \xi \in I_{C_{R_1}(\frac{1}{2}), M_-, M_+, 1} \quad (49h)$$

$$\text{sign}(\alpha_{R_1, M_-, M_+ - 1, -M_-}(\xi)) \stackrel{(49f)}{=} (-1)^{M_- + M_+ - 1} \stackrel{(2a)}{=} (-1)^{M_-} \quad \forall \xi \in I_{C_{R_1}(\frac{1}{2}), M_-, M_+, 1} \quad (49i)$$

$$\begin{aligned} \text{sign}(\alpha_{R_1, M_- - 1, M_+, +M_+}(\xi)) &\stackrel{(27a)}{=} (-1)^{M_- - 1} \text{sign}(\alpha_{R_1, M_-, M_+ - 1, -M_-}(\xi)) \\ &\stackrel{(49f)}{=} (-1)^{M_- + M_+} = (-1)^{M_+ - 1} \quad \forall \xi \in I_{C_{R_1}(\frac{1}{2}), M_-, M_+, 1} \end{aligned} \quad (49j)$$

whence

$$\sigma_{R_1, M_-, M_+, 1, 0}(\xi) \stackrel{(32a, 49g, 49i)}{>} 0 \quad \forall \xi \in I_{C_{R_1}(\frac{1}{2}), M_-, M_+, 1} \quad (49k)$$

$$\sigma_{R_1, M_-, M_+, 1, 1}(\xi) \stackrel{(32b, 49h, 49j)}{>} 0 \quad \forall \xi \in I_{C_{R_1}(\frac{1}{2}), M_-, M_+, 1} \quad (49l)$$

Because of the consistency condition (32c), positivity of the weight-functions implies convexity (Remark 4.10), so that (49k, 49l) prove (48a). Notice that, by Proposition 3.5, (24a) implies that $\xi_{R_1, M_-, M_+, \ell \neq 0, 0} < \frac{1}{2}$ and $\xi_{R_1, M_-, M_+, \ell \neq 1, 1} > \frac{1}{2}$, $\forall M_{\pm} \in \mathbb{Z} : M := M_- + M_+ > 2$, satisfying the conditions of Corollary 4.12, so that the length of $I_{C_{R_1}(\frac{1}{2}), M_-, M_+, 1}$ (48a–48c) is > 0 . \square

Theorem 4.14 (Convexity of a positive subdivision in the neighborhood of $i + \frac{1}{2}$). *Assume that the subdivision level $K_s \geq 1$ of S_{i, M_-, M_+} (Definition 1.2) satisfies the conditions of Lemma 4.11 (positive subdivision), viz*

$$M := M_- + M_+ \geq 2 \quad (2a)$$

$$-M_- \leq 0 < 1 \leq M_+ \quad (46a)$$

$$1 \leq K_s \leq \min(M_- + 1, M_+) \quad (46b)$$

implying (Lemma 4.11) that all substencils contain either point i or point $i + 1$. Define the interval

$$I_{C_{R_1}(\frac{1}{2}), M_-, M_+, K_s} := \bigcap_{L_s=0}^{K_s-1} \bigcap_{\ell_s=0}^{L_s} I_{C_{R_1}(\frac{1}{2}), M_- - \ell_s, M_+ - L_s + \ell_s, 1} \quad (50a)$$

recursively using convexity intervals $I_{C_{R_1}(\frac{1}{2}), M_-, M_+, 1}$ (48a–48c) of $(K_s = 1)$ -level positive subdivisions (Lemma 4.13). Then the rational weight-functions $\sigma_{R_1, M_-, M_+, K_s, k_s}(\xi)$ (39b) satisfy

$$0 < \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) < 1 \quad \begin{cases} \forall \xi \in I_{C_{R_1}(\frac{1}{2}), M_-, M_+, K_s} \\ \forall K_s \in \{1, \dots, M - 1\} \\ \forall k_s \in \{0, K_s\} \end{cases} \quad (50b)$$

implying convexity of the combination (39a).

PROOF. The validity of (50a, 50b) for $K_s = 1$ was proven in Lemma 4.13. Assume $\min(M_- + 1, M_+) \geq 2$ so that the $(K_s = 2)$ -level subdivision be a positive subdivision (Lemma 4.11). Then, by (39b), we have

$$\sigma_{R_1, M_-, M_+, 2, 0}(\xi) \stackrel{(39b)}{=} \sigma_{R_1, M_-, M_+, 1, 0}(\xi) \sigma_{R_1, M_-, M_+ - 1, 1, 0}(\xi) \quad (51a)$$

$$\begin{aligned} \sigma_{R_1, M_-, M_+, 2, 1}(\xi) &\stackrel{(39b)}{=} \sigma_{R_1, M_-, M_+, 1, 0}(\xi) \sigma_{R_1, M_-, M_+ - 1, 1, 1}(\xi) \\ &\quad + \sigma_{R_1, M_-, M_+, 1, 1}(\xi) \sigma_{R_1, M_- - 1, M_+, 1, 0}(\xi) \end{aligned} \quad (51b)$$

$$\sigma_{R_1, M_-, M_+, 2, 2}(\xi) \stackrel{(39b)}{=} \sigma_{R_1, M_-, M_+, 1, 1}(\xi) \sigma_{R_1, M_- - 1, M_+, 1, 1}(\xi) \quad (51c)$$

Having assumed that the $(K_s = 2)$ -level subdivision is a positive subdivision (Lemma 4.11), we have

$$\min(M_- + 1, M_+) \geq 2 \implies \left[\begin{array}{l} M_- + 1 \geq 2 \xRightarrow{(2a)} -(M_- - 1) \leq 0 < 1 \leq M_+ \\ M_+ \geq 2 \xRightarrow{(2a)} -M_- \leq 0 < 1 \leq (M_+ - 1) \end{array} \right] \quad (51d)$$

implying that the 1-level subdivisions of the stencils $S_{i,M_- - 1, M_+}$ and $S_{i,M_-, M_+ - 1}$ are positive (Corollary 4.12). Therefore all of the 1-level weight-functions on the RHS of (51a–51c) are positive in the neighborhood of $\xi = \frac{1}{2}$, because of Lemma 4.13, and we have

$$0 < \sigma_{R_1, M_-, M_+, 2, 0}(\xi) \quad \forall \xi \in \left(I_{C_{R_1}(\frac{1}{2}), M_-, M_+, 1} \cap I_{C_{R_1}(\frac{1}{2}), M_-, M_+ - 1, 1} \right) \quad (51e)$$

$$0 < \sigma_{R_1, M_-, M_+, 2, 1}(\xi) \quad \forall \xi \in \left(I_{C_{R_1}(\frac{1}{2}), M_-, M_+, 1} \cap I_{C_{R_1}(\frac{1}{2}), M_-, M_+ - 1, 1} \cap I_{C_{R_1}(\frac{1}{2}), M_- - 1, M_+, 1} \right) \quad (51f)$$

$$0 < \sigma_{R_1, M_-, M_+, 2, 2}(\xi) \quad \forall \xi \in \left(I_{C_{R_1}(\frac{1}{2}), M_-, M_+, 1} \cap I_{C_{R_1}(\frac{1}{2}), M_- - 1, M_+, 1} \right) \quad (51g)$$

Defining

$$I_{C_{R_1}(\frac{1}{2}), M_-, M_+, 2} \stackrel{(50a)}{:=} \bigcap_{L_s=0}^1 \bigcap_{\ell_s=0}^{L_s} I_{C_{R_1}(\frac{1}{2}), M_-, M_+ - L_s + \ell_s, 1} = I_{C_{R_1}(\frac{1}{2}), M_-, M_+, 1} \cap I_{C_{R_1}(\frac{1}{2}), M_-, M_+ - 1, 1} \cap I_{C_{R_1}(\frac{1}{2}), M_- - 1, M_+, 1} \quad (51h)$$

we have that all of the 3 $(K_s = 2)$ -level weight-functions (51e–51g) are simultaneously positive $\forall \xi \in I_{C_{R_1}(\frac{1}{2}), M_-, M_+, 2}$, which (Remark 4.10), because of the consistency condition (39b), proves (50a, 50b) for $K_s = 2$.

It is straightforward to complete the proof by induction. Since we have already proved (50a, 50b) for $K_s = 2$, assume $K_s - 1 \geq 2 \iff K_s \geq 3$. By Proposition 4.5

$$\sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) \stackrel{(39b)}{=} \sum_{\ell_s=\max(0, k_s-1)}^{\min(K_s-1, k_s)} \sigma_{R_1, M_-, M_+, K_s-1, \ell_s}(\xi) \sigma_{R_1, M_-, M_+ - (K_s-1) + \ell_s, 1, k_s - \ell_s}(\xi) \quad \forall k_s \in \{0, \dots, K_s\} \quad (51i)$$

Assume that (50a, 50b) are valid for $K_s - 1 \geq 2$

$$0 < \sigma_{R_1, M_-, M_+, K_s-1, \ell_s}(\xi) < 1 \quad \left\{ \begin{array}{l} \forall \xi \in I_{C_{R_1}(\frac{1}{2}), M_-, M_+, K_s-1} = \bigcap_{L_s=0}^{K_s-2} \bigcap_{\ell_s=0}^{L_s} I_{C_{R_1}(\frac{1}{2}), M_-, M_+ - L_s + \ell_s, 1} \\ \forall \ell_s \in \{0, K_s - 1\} \end{array} \right. \quad (51j)$$

and that K_s satisfies (46b), ie

$$\min(M_- + 1, M_+) \geq K_s \implies \left[\begin{array}{l} M_- + 1 \geq K_s \\ M_+ \geq K_s \end{array} \right] \xRightarrow{(2a)} -(M_- - \ell_s) \leq 0 < 1 \leq (M_+ - (K_s - 1) + \ell_s) \quad \forall \ell_s \in \{0, \dots, K_s - 1\} \quad (51k)$$

so that the substencils $S_{i, M_- - \ell_s, M_+ - (K_s - 1) + \ell_s}$ satisfy the conditions of Corollary 4.12, implying by Lemma 4.13 that

$$0 < \sigma_{R_1, M_-, M_+ - (K_s - 1) + \ell_s, 1, m_s}(\xi) \quad \forall \xi \in I_{C_{R_1}(\frac{1}{2}), M_-, M_+ - (K_s - 1) + \ell_s, 1} \quad \left\{ \begin{array}{l} \forall m_s \in \{0, 1\} \\ \forall \ell_s \in \{0, \dots, K_s - 1\} \end{array} \right. \quad (51l)$$

Combining (51i, 51j, 51l) yields

$$0 < \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) \quad \left\{ \begin{array}{l} \forall \xi \in I_{C_{R_1}(\frac{1}{2}), M_-, M_+, K_s-1} \cap \left(\bigcap_{\ell_s=0}^{K_s-1} I_{C_{R_1}(\frac{1}{2}), M_-, M_+ - (K_s-1) + \ell_s, 1} \right) \\ \forall k_s \in \{0, K_s\} \end{array} \right. \quad (51m)$$

which (Remark 4.10), because of the consistency condition (39b), proves (50a, 50b) $\forall K_s$ satisfying (46b). \square

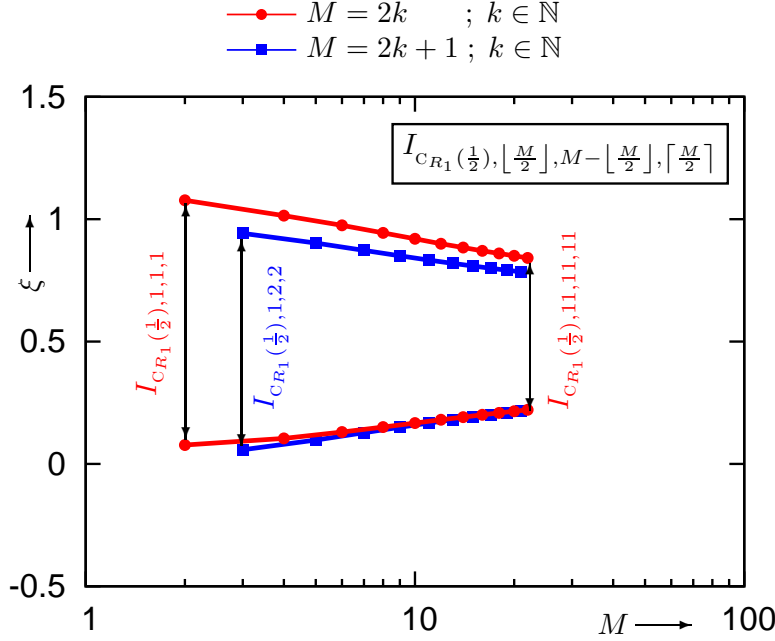


Figure 7: Interval of convexity $I_{C_{R_1}(\frac{1}{2}), \lfloor \frac{M}{2} \rfloor, M - \lfloor \frac{M}{2} \rfloor, \lceil \frac{M}{2} \rceil}$ around $i + \frac{1}{2}$ (Theorem 4.14), of the maximum positive subdivision-level (Lemma 4.11) $K_s = \lceil \frac{M}{2} \rceil$, of the usual WENO stencils $S_{i, \lfloor \frac{M}{2} \rfloor, M - \lfloor \frac{M}{2} \rfloor}$ (Definition 1.1), as a function of stencil width M (in logscale).

Example 4.15 (Convexity around $i + \frac{1}{2}$ of usual WENO discretizations). The usual WENO discretizations for the numerical approximation of $f'(x)$ [5, 3, 11] use [7, p. 298] the $(K_s = \lceil \frac{M}{2} \rceil)$ -level subdivision (Definition 1.2) of the general family of stencils $S_{i, \lfloor \frac{M}{2} \rfloor, M - \lfloor \frac{M}{2} \rfloor}$, which (Definition 1.1) contains $M + 1$ points. If $M = 2k$ ($k \in \mathbb{N}_{>0}$) is even, then the stencil $S_{i, \lfloor \frac{M}{2} \rfloor, M - \lfloor \frac{M}{2} \rfloor} = S_{i, k, k}$ is symmetric around point i , and upwind-biased with respect to the cell-interface $i + \frac{1}{2}$ (eg $S_{i, 3, 3}$; Fig. 5), corresponding to the family of WENO($2r - 1$) ($r := k + 1$) upwind-biased schemes [2, 9, 10]. If $M = 2k + 1$ ($k \in \mathbb{N}_{>0}$) is odd, then the stencil $S_{i, \lfloor \frac{M}{2} \rfloor, M - \lfloor \frac{M}{2} \rfloor} = S_{i, k, k+1}$ is symmetric around the cell-interface $i + \frac{1}{2}$, and downwind-biased with respect to the point i (eg $S_{i, 3, 4}$; Fig. 6), corresponding to centered (central) WENO schemes [15]. For the family of stencils $S_{i, \lfloor \frac{M}{2} \rfloor, M - \lfloor \frac{M}{2} \rfloor}$, we have

$$M_- = \left\lfloor \frac{M}{2} \right\rfloor \quad (52a)$$

$$M_+ = M - \left\lfloor \frac{M}{2} \right\rfloor \quad (52b)$$

$$\min(M_- + 1, M_+) = \begin{cases} \min(k + 1, k) = k & \forall M = 2k \quad ; k \in \mathbb{N}_{>0} \\ \min(k + 1, k + 1) = k + 1 & \forall M = 2k + 1 \quad ; k \in \mathbb{N}_{>0} \end{cases} = \left\lceil \frac{M}{2} \right\rceil \quad \forall M \in \mathbb{N}_{\geq 2} \quad (52c)$$

so that, by Lemma 4.11, $K_s = \min(\lfloor \frac{M}{2} \rfloor + 1, M - \lfloor \frac{M}{2} \rfloor) \stackrel{(52)}{=} \lceil \frac{M}{2} \rceil$ corresponds to the maximum level of positive subdivision of $S_{i, \lfloor \frac{M}{2} \rfloor, M - \lfloor \frac{M}{2} \rfloor}$. Therefore, Theorem 4.14 applies, and there exists an interval of convexity $I_{C_{R_1}(\frac{1}{2}), \lfloor \frac{M}{2} \rfloor, M - \lfloor \frac{M}{2} \rfloor, \lceil \frac{M}{2} \rceil}$ (50a) around $\xi = \frac{1}{2}$ (Fig. 7).

Notice that the interval of convexity $I_{C_{R_1}(\frac{1}{2}), \lfloor \frac{M}{2} \rfloor, M - \lfloor \frac{M}{2} \rfloor, \lceil \frac{M}{2} \rceil}$ (Fig. 7), is slightly larger for $M = 2k$ ($k \in \mathbb{N}_{>0}$) even, compared to $M = 2k + 1$ ($k \in \mathbb{N}_{>0}$) odd, and its length slightly decreases (quasi-logarithmically $\forall M \in \{2, \dots, 22\}$) with increasing number of cells in the stencil, M (Fig. 7). For stencils with $M = 2k$ ($k \in \mathbb{N}_{>0}$) even, like $S_{i, 3, 3}$ (Fig. 3), because of symmetry with respect to point i , it is straightforward to show that there is a symmetric interval of convexity around $i - \frac{1}{2}$ (Fig. 5), as was also observed in [11, Tab. 3.2, p. 516]. On the contrary, for stencils with $M = 2k + 1$ ($k \in \mathbb{N}_{>0}$) odd, like $S_{i, 3, 4}$ (Fig. 4), it turns out that positivity of the weight-functions does not hold at $\xi = -\frac{1}{2}$ (Example 4.9; Fig. 6), as was also observed in [11, Tab. 3.5, p. 518]. \square

5. Conclusions

In the present work, we studied analytically the representation of the Lagrange reconstructing polynomial by combination of substencils, and in particular the conditions under which this representation is convex, *ie* the weight-functions $\in [0, 1]$.

We first formalized several results on the fundamental polynomials of Lagrange reconstruction (Proposition 2.2), $\alpha_{R_1, M_-, M_+, \ell}(\xi)$ (11a). Each of the polynomials $\alpha_{R_1, M_-, M_+, \ell}(\xi)$ is the reconstruction pair (Definition 1.3) of the corresponding fundamental function of Lagrange interpolation $\alpha_{I, M_-, M_+, \ell}(\xi)$ (Proposition 3.1), and for this reason all of its M roots are real (Proposition 3.5), distant $< \frac{1}{2}$ from the corresponding root of the fundamental function of Lagrange interpolation $\alpha_{I, M_-, M_+, \ell}(\xi)$ (12a). This leads to a simple factorization of the fundamental polynomials of Lagrange reconstruction (Proposition 3.7).

The leading $O(\Delta x^M)$ term of the approximation error of the Lagrange reconstructing polynomials on 2 overlapping stencils shifted by 1 cell, $\{i - M_-, \dots, i + M_+ - 1\}$ and $\{i - M_- + 1, \dots, i + M_+\}$, is different (Proposition 3.10), and several identities hold between some of the fundamental polynomials on the 2 stencils. Based on these identities (Proposition 3.10), we show that there exist unique rational weight-functions combining the Lagrange reconstructing polynomials on $\{i - M_-, \dots, i + M_+ - 1\}$ and $\{i - M_- + 1, \dots, i + M_+\}$ into the Lagrange reconstructing polynomials on $\{i - M_-, \dots, i + M_+\}$ (Lemma 4.2), this representation failing at the poles of the weight-functions, all of which are real and can be identified with roots of fundamental polynomials of Lagrange reconstruction. Having established this 1-level subdivision rule, the general recurrence relation for the weight-functions proven in [12, Lemma 2.1] applies, and provides the analytical expression of the weight-functions for a general level of subdivision (Proposition 4.5), and of the set of their poles, all of which are real. These weight-functions are unique (Proposition 4.7).

Finally, we prove (Theorem 4.14) that for any K_s -level subdivision of $\{i - M_-, \dots, i + M_+\}$ into $K_s + 1$ substencils $\{i - M_- + k_s, \dots, M_+ - K_s + k_s\}$ ($k_s \in \{0, \dots, K_s\}$), iff each of the substencils contains either point i or point $i + 1$ (positive subdivision; Lemma 4.11), then there exists a neighborhood of $\xi = \frac{1}{2}$ ($x = x_i + \frac{1}{2}\Delta x$), $I_{C_{R_1}(\frac{1}{2}), M_-, M_+, K_s} \ni \frac{1}{2}$, whose limits can be explicitly defined by roots of fundamental polynomials of Lagrange reconstruction, where all of the weight-functions $\sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) > 0 \ \forall \xi \in I_{C_{R_1}(\frac{1}{2}), M_-, M_+, K_s}$, implying because of the consistency relation $\sum_{k_s=0}^{K_s} \sigma_{R_1, M_-, M_+, K_s, k_s}(\xi) = 1 \ \forall \xi \in \mathbb{R}$ (39c), that the representation of the Lagrange reconstructing polynomial by combination of substencils is convex $\forall \xi \in I_{C_{R_1}(\frac{1}{2}), M_-, M_+, K_s} \ni \frac{1}{2}$. Theorem 4.14 provides a formal proof of (and general conditions for) convexity in the neighborhood of $\xi = \frac{1}{2}$, which had always been conjectured, on the basis of numerical evidence, all along the development of WENO schemes [1, 2, 5, 9, 3, 11, 10].

Acknowledgments

Computations were performed using HPC resources from GENCI-IDRIS (Grants 2010–066327 and 2010–022139). Symbolic calculations were performed using maxima (<http://sourceforge.net/projects/maxima>). The corresponding package `reconstruction.mac` is available at <http://www.aerodynamics.fr>.

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